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IFCRL: Interval-intent Fuzzy Concept Re-cognition Learning Model

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***Abstract*—The fuzzy concept serves as a crucial tool for describing phenomena and constitutes the fundamental unit of human cognition. Fuzzy concepts are characterized by their extent and intent, with the latter being comprised of continuous membership degrees. Given that human cognition often progresses from vagueness to precision, it is imperative that the form of intent not be confined to a singular continuous value; rather, an interval possesses superior flexibility in this regard. Initial cognitive processes lack comprehensiveness in acquiring knowledge, necessitating subsequent cognitions to more accurately delineate the intended scope of a concept. Motivated by this insight, we proposed an interval-intent fuzzy concept re-cognition learning model (IFCRL). Firstly, this model transforms fuzzy concept intent from a single continuous value into an interval-based representation which describes the range of attribute values for all objects within the given interval. Secondly, in order to simulate secondary cognitive processes akin to those exhibited by humans towards phenomena, we present a concept re-cognition learning method capable of effectively scaling intervals within reasonable bounds. Thirdly, aiming to overcome cognitive barriers and emulate imaginative processes observed in human brains, we introduce a concept clustering approach based on intent similarity which significantly reduces concept complexity while enhancing cognitive efficiency. Finally, we evaluate our classification performance using 12 datasets and experimental results demonstrate that IFCRL outperforms 14 other classification algorithms both feasibly and effectively.**

***Index Terms*—Concept-cognitive learning, Concept clustering, Granular computing, Interval-intent, Object classification.**

I. Introduction

COGNITIVE informatics is an emerging interdisciplinary research field that integrates various domains, including modern informatics, artificial intelligence, cybernetics, cognitive science, neuropsychology, medical science, philosophy, linguistics, life sciences and others [1]. In the realm of cognitive informatics, relations are recognized as information. The connections between objects can be established through object-object relationships, attribute-object relationships or attribute-attribute relationships [2]. As the fundamental unit of human cognition, concepts encompass the relationship between objects and attributes. The extent of a concept refers to the set of all objects or instances that the concept represents, while the intent of a concept pertains to the set of attributes or properties that it implies [3]. Investigating the relationship between entities through the integration of concepts with mathematical, psychological, and other methodologies has emerged as a prominent research direction. Moreover, concept learning has expanded into various interconnected research domains including granular computing [4], [5], [6], [7], rough set [8], [9], [10], formal concept analysis [11], [12], [13], [14], among others.

The field of concept-cognitive learning encompasses the study of cognition and the acquisition of knowledge through conceptual frameworks [15]. In recent years, many different types of concept learning models have been proposed, such as abstract concept [16], Wille's formal concept [17], object-oriented concept [18], fuzzy concept [19], [20], [21], [22], three-way concept [23], [24], two-way concept [25], [26], [27]. In terms of basic theory, the framework of concept learning was investigated by Yao [28], taking into account both cognitive science and granular computing. Zhang et al. [29] analyzed the sufficient and necessary between attributes and objects, and combined intuition and reasoning to establish a rigorous mathematical model to simulate human cognitive processes. Based on this, Xu et al. [30] extensively discussed the theory of transforming arbitrary information granules into necessary and sufficient information granules. Zhang et al. [31] proposed a two-way concept-cognitive learning model based on three-way decision under fuzzy context, which can directly learn sufficient and necessary concepts from arbitrary information granules. Additionally, Xu et al. [25] proposed a two-way dynamic concept-cognitive learning model within a fuzzy context.

The theoretical system of two-way concept-cognitive learning has been gradually complete. At the same time, the model combining concept-cognitive with machine learning has developed rapidly in recent years. A perspective on machine learning was presented by Mi et al. [32], introducing a comprehensive approach to cognitive learning. For application to classification problems, many concept-cognitive learning models [33], [34], [35], [36] have been proposed. Shi et al. [33] proposed a concept cognitive learning model that is good at incremental learning to implement static and dynamic classification tasks. Mi et al. [34] proposed a concept clustering method considering object information in fuzzy context and applied it to the problem of concept generation. Niu et al. [37] proposed a classification model based on fuzzy rules, which realizes granularity reduction and dynamic update in fuzzy environment. Liu et al. [38] proposed a stochastic incremental

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incomplete concept-cognitive learning method that is not influenced by the cognitive sequence of attributes, and employed a stochastic strategy for cognition. Hu et al. [39] explored concept learning models in the interval-valued context. Zhang et al. [40] proposed a incremental weight concept-cognitive learning algorithm based on fuzzy entropy to solve individual cognitive limitation.

The existing concept-cognitive learning still holds immense potential for development. However, the current classification learning models fail to consider interval-valued formal concepts in a fuzzy context. Each granular concept is learned only once, resulting in relatively shallow acquired knowledge. Additionally, the existing concept cognitive learning models restrict cognitive thinking and create cognitive barriers by considering object information as the basis for participation in clustering processes (concepts can only be clustered if their extent intersect). To delve deeper into knowledge, obtain more accurate concepts, and break down cognitive barriers, we propose an interval-intent fuzzy concept re-cognition learning model under a fuzzy context. The primary contributions of this paper are as follows:

1) We introduce new mapping operators that form the basis of a novel concept definition method called interval-intent fuzzy concept. We also discuss the fundamental properties of interval-intent fuzzy concepts and define two types of interval-intent fuzzy granular concepts based on granular computing to represent data from different perspectives.

2) Building upon these new granular concepts, we propose a concept re-cognition process that effectively reduces cognitive errors caused by noise data (mainly refers to extreme value cases) and closely mimics human secondary cognition.

3) Furthermore, the proposed clustering algorithm is based on intent similarity, effectively overcoming the cognitive barrier (concepts can only be clustered if their extent intersect). This approach significantly reduces the concept space and enhances classification efficiency.

4) Finally, we compare IFCRL with fourteen different classification algorithms using twelve datasets from UCI and KEEL databases. Moreover, we analyze how parameters influence both the size and accuracy of the clustering space. Our results demonstrate that IFCRL achieves superior average accuracy compared to other methods.

Compared to precise fuzzy values, interval values offer greater flexibility and comprehensibility in our cognitive process. The concept's intent shifts from fuzzy values to interval values, allowing for a reasonable summarization of the attribute membership degrees of all objects within the scope. The re-cognition process emulates humans' second understanding of things, eliminating cognitive errors caused by noise and enabling a more accurate comprehension of the concept's intent. Clustering based on intent similarity transcends cognitive boundaries and presents new possibilities for cognition (even extent-disjoint concepts can be clustered).

The subsequent sections of this paper are structured as follows: Section II introduces the interval set correlation operations used in this study along with basic definitions of fuzzy concepts while discussing the motivation behind this research. Section III proposes a model for interval-intent fuzzy concept re-cognition learning. Extensive experiments are conducted in Section IV to validate the feasibility of the proposed model. Finally, we conclude this work and provide suggestions for future research.

II. Related work

This section mainly introduces the related operations of interval sets and the relevant knowledge of fuzzy concept-cognitive learning.

A. Interval Set

Let U stands for the unit closed interval from 0 to 1, and $[U]$ denote the set of all closed intervals on the interval [0,1]. If M be a nonempty set, we call the mapping $\tilde{B}: M \to [U]$ an interval set on M. And all interval sets on M are denoted as $I(M)$.

For $\forall \tilde{B} \in I(M)$, let $\tilde{B}^{\pm}(c) = [\tilde{B}^{-}(c), \tilde{B}^{+}(c)], c \in M$. We define ordinary fuzzy sets $\tilde{B}^{-}, \tilde{B}^{+} : M \to U$ as lower-interval sets and upper-interval sets of M, respectively. For convenience, $\tilde{B}^{\pm}$ is used to represent an interval set $[\tilde{B}^{-}, \tilde{B}^{+}]_{\forall c}$ in this paper. $\tilde{B}^{-}$ and $\tilde{B}^{+}$ denote the ordered set consisting of all lower and upper bounds of the interval set $\tilde{B}^{\pm}$.

The four representations of interval sets in this paper are as follows:

$$\begin{aligned}&\tilde{B}^{\pm} = [\tilde{B}^{-}, \tilde{B}^{+}]_{\forall c}\\&= [\{\tilde{B}^{-}(c_1), \cdots, \tilde{B}^{-}(c_{|M|})\}, \{\tilde{B}^{+}(c_1), \cdots, \tilde{B}^{+}(c_{|M|})\}]_{\forall c}\\&= \{[\tilde{B}^{-}(c_1), \tilde{B}^{+}(c_1)], \cdots, [\tilde{B}^{-}(c_{|M|}), \tilde{B}^{+}(c_{|M|})]\}\end{aligned}$$

The interval inclusion relationship consistent with human cognition is defined as $\tilde{B}_1^{\pm} \subseteq \tilde{B}_2^{\pm}$ if $\forall c \in M, \tilde{B}_1^{-}(c) \geq \tilde{B}_2^{-}(c)$ and $\tilde{B}_1^{+}(c) \leq \tilde{B}_2^{+}(c)$. We call two interval sets $\tilde{B}_1^{\pm}$ and $\tilde{B}_2^{\pm}$ are intersected if the following conditions are satisfied: $\exists a_i \in U$, $\tilde{B}_1^{-}(c_i) \leq a_i \leq \tilde{B}_1^{+}(c_i)$ and $\tilde{B}_2^{-}(c_i) \leq a_i \leq \tilde{B}_2^{+}(c_i)$, $i =1, 2, \cdots, |M|$.

Definition 1. Given two interval sets $\tilde{B}_1^{\pm}$ and $\tilde{B}_2^{\pm}$ which are intersected, their intersection and union operations are described as the following formula:

$$\begin{aligned}\tilde{B}_1^{\pm} \cap \tilde{B}_2^{\pm} =&[\{\max(\tilde{B}_1^{-}(c_i), \tilde{B}_2^{-}(c_i)) \mid i = 1, 2, \cdots, |M|\},\\&\{\min(\tilde{B}_1^{+}(c_i), \tilde{B}_2^{+}(c_i)) \mid i = 1, 2, \cdots, |M|\}]_{\forall c};\\\tilde{B}_1^{\pm} \cup \tilde{B}_2^{\pm} =&[\{\min(\tilde{B}_1^{-}(c_i), \tilde{B}_2^{-}(c_i)) \mid i = 1, 2, \cdots, |M|\},\\&\{\max(\tilde{B}_1^{+}(c_i), \tilde{B}_2^{+}(c_i)) \mid i = 1, 2, \cdots, |M|\}]_{\forall c}.\end{aligned}$$

We define the number multiplication and addition operations of the interval set as follows:

$$a \cdot \tilde{B}_1^{\pm}(c) = [a \cdot \tilde{B}_1^{-}(c), a \cdot \tilde{B}_1^{+}(c)]_{\forall c}$$

$$\tilde{B}_1^{\pm}(c) + \tilde{B}_2^{\pm}(c) = [\tilde{B}_1^{-}(c) + \tilde{B}_2^{-}(c), \tilde{B}_1^{+}(c) + \tilde{B}_2^{+}(c)]_{\forall c}$$

B. Fuzzy Concept-cognitive Learning

Fuzzy formal context $(G, M, \tilde{I})$ is a triple, where G represents the object set, M represents the attribute set, and $\tilde{I}$ represents the fuzzy relationship between the object and the attribute. The membership degree $\tilde{I}(x, c)$ between object x and

TABLE I
A FUZZY FORMAL DECISION CONTEXT.

G	c_1	c_2	d	G	c_1	c_2	d
x_1	0.32	0.63	1	x_7	0.80	0.70	2
x_2	0.36	0.52	1	x_8	0.81	0.66	2
x_3	0.48	0.83	1	x_9	0.77	0.60	2
x_4	0.38	0.48	1	x_{10}	0.91	0.55	2
x_5	0.40	0.60	1	x_{11}	0.72	0.45	2
x_6	0.41	0.76	1	x_{12}	0.58	0.61	2

attribute c in $\tilde{I}$ satisfied $\tilde{I}(x,c) \in [0,1]$. We call the quintuple $(G, M, \tilde{I}, D, J)$ fuzzy-classical decision formal context, where $\tilde{I} : G \times M \to [0,1]$ and $J : G \times D \to \{0,1\}$.

Let $(G, M, \tilde{I})$ be a fuzzy formal context. For $X \subseteq G$ and $\tilde{B} \in I(M)$, the two mapping operators $\tilde{L}$ and N are defined as follows [41], [42]:

$$\tilde{L}(X) = \{ \bigwedge_{x\in X} \tilde{I}(x,c_i) | i = 1,2,\cdots,|M|\},$$

$$N(\tilde{B}) = \{x \in G | \forall c_i \in M, \tilde{B}(c_i) \leq \tilde{I}(x,c_i)\}.$$

We call a pair $(X, \tilde{B})$ the fuzzy concept if $\tilde{L}(X) = \tilde{B}$ and $N(\tilde{B}) = X$. And we call X the extent of the concept and $\tilde{B}$ the intent of the concept. Generally speaking, the learning process is complicated due to the large number of fuzzy concepts. Therefore, the idea of particle computing is introduced to simplify it into fuzzy granular concept. The fuzzy conditional granular concept derived from any object x is defined as $(N\tilde{L}(x), \tilde{L}(x))$.

Example 1. The table I represents a fuzzy formal decision context consisting of twelve objects and two condition attributes, where $X_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $X_2 = \{x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$. For the case of decision $d = 1$, the classical fuzzy conditional granular concepts are $(\{x_1, x_3, x_6\}, \{0.32, 0.63\})$, $(\{x_2, x_3, x_5, x_6\}, \{0.36, 0.52\})$, $(\{x_3\}, \{0.48, 0.83\})$, $(\{x_3, x_4, x_5, x_6\}, \{0.38, 0.48\})$, $(\{x_3, x_5, x_6\}, \{0.4, 0.6\})$, $(\{x_3, x_6\}, \{0.41, 0.76\})$.

C. Motivation

The classical formal concepts primarily pertains to the depiction of the association between discrete data objects and their attributes. However, many real-world datasets are not binary or discrete, resulting in significant information loss during processing. Fuzzy concepts offer a viable solution to these issues. Nevertheless, existing fuzzy concepts typically represent a single value that accurately describes the lower bound of a concept's membership degree. Human perception often relies on a two-way scope for defining things rather than a one-way approach. For instance, gray is considered as being between black and white without being darker than white or whiter than black.

The unidirectional nature of fuzzy concepts fails to adequately describe objects with membership degrees close to the intent (slightly higher or slightly lower). To address this limitation, this paper proposes an interval-intent fuzzy concept that expands upon the intent of fuzzy concepts by adopting an interval form. This effectively characterizes concepts with two boundaries instead of just one boundary as seen in traditional fuzzy concepts. An interval inherently possesses upper and lower bounds, which aligns with the notion of considering concepts from both directions simultaneously. Moreover, intervals correspond well with human cognitive habits since attribute values tend to cluster among similar entities.

III. INTERVAL-INTENT FUZZY CONCEPT RE-COGNITION LEARNING MODEL

In this section, we introduce the definition and related properties of a new interval-intent fuzzy concept, and propose the process of concept re-cognition and a new method of concept clustering.

A. Interval-intent Fuzzy Concept

Definition 2. For a fuzzy formal context $(G, M, \tilde{I})$, we define four mapping operators $\tilde{F}^-, \tilde{F}^+ : 2^G \to [L^M]$ and $H^-, H^+ : [L^M] \to 2^G$ as follows:

$$\tilde{F}^-(X) = \{[\bigwedge_{x\in X} \tilde{I}(x,c_i), 1] | i = 1,2,\cdots,|M|\},$$

$$\tilde{F}^+(X) = \{[0, \bigvee_{x\in X} \tilde{I}(x,c_i)] | i = 1,2,\cdots,|M|\},$$

$$H^-(\tilde{B}^\pm) = \{x \in G | \tilde{B}^-(c) \leq \tilde{I}(x,c), \forall c \in M\},$$

$$H^+(\tilde{B}^\pm) = \{x \in G | \tilde{B}^+(c) \geq \tilde{I}(x,c), \forall c \in M\}.$$

where 2^G represents the power set of the object set and $[L^M]$ represents the power set of the attribute interval set.

According to the theory of the previous section, it is easy to prove $\tilde{F}^-(X)$ and $\tilde{F}^+(X)$ are intersected. It is feasible and concise to merge the four mapping operators defined above into two.

Definition 3. Given the four mapping operators in Definition 2, we define the two merged mapping operators $\tilde{F}^\pm : 2^G \to [L^M], H^\pm : [L^M] \to 2^G$ as follows:

$$\begin{aligned}\tilde{F}^\pm(X) &= \tilde{F}^-(X) \cap \tilde{F}^+(X) \\ &= \{[\bigwedge_{x\in X} \tilde{I}(x,c_i), \bigvee_{x\in X} \tilde{I}(x,c_i)] | i = 1,2,\cdots,|M|\}, \\ H^\pm(\tilde{B}^\pm) &= H^+(\tilde{B}^\pm) \cap H^-(\tilde{B}^\pm) \\ &= \{x \in G | \tilde{B}^-(c) \leq \tilde{I}(x,c) \leq \tilde{B}^+(c), \forall c \in M\}.\end{aligned}$$

If $\tilde{F}^\pm(X) = \tilde{B}^\pm$ and $H^\pm(\tilde{B}^\pm) = X$, we call an ordered pair $(X, \tilde{B}^\pm)$ an interval-intent fuzzy concept. We call a super-concept $(X_2, \tilde{B}_2^\pm)$ and a subconcept $(X_1, \tilde{B}_1^\pm)$ have the order relation $(X_1, \tilde{B}_1^\pm) \leq (X_2, \tilde{B}_2^\pm)$ if $X_1 \subseteq X_2$ (or $\tilde{B}_1^\pm \subseteq \tilde{B}_2^\pm$). If there is no confusion, we will rewrite the first merged mapping to form $\tilde{F}^\pm(X) = [\bigwedge_{x\in X} \tilde{I}(x,c), \bigvee_{x\in X} \tilde{I}(x,c)]_{\forall c}$.

Property 1. Let $X, X_1, X_2 \subseteq G, \tilde{B}^\pm, \tilde{B}_1^\pm, B_2^\pm \subseteq [L^M]$, then we have:

(1) $\tilde{F}^\pm(X_1) \subseteq \tilde{F}^\pm(X_2)$ if $X_1 \subseteq X_2$, $H^\pm(\tilde{B}_1^\pm) \subseteq H^\pm(\tilde{B}_2^\pm)$ if $\tilde{B}_1^\pm \subseteq \tilde{B}_2^\pm$;
(2) $X \subseteq H^\pm \tilde{F}^\pm(X)$, $\tilde{B}^\pm \supseteq \tilde{F}^\pm H^\pm(\tilde{B}^\pm)$;
(3) $\tilde{F}^\pm(X) = \tilde{F}^\pm H^\pm \tilde{F}^\pm(X)$ and $H^\pm(\tilde{B}^\pm) = H^\pm \tilde{F}^\pm H^\pm(\tilde{B}^\pm)$.

Proof. (1) According to the definition of the object to property mapping operator, we have $\tilde{F}^{\pm}(X_1) = [\bigwedge_{x\in X_1} \tilde{I}(x,c), \bigvee_{x\in X_1} \tilde{I}(x,c)]_{\forall c}$ and $\tilde{F}^{\pm}(X_2) = [\bigwedge_{x\in X_2} \tilde{I}(x,c), \bigvee_{x\in X_2} \tilde{I}(x,c)]_{\forall c}$. Because $X_1 \subseteq X_2$, $\bigwedge_{x\in X_1} \tilde{I}(x,c) \geq \bigwedge_{x\in X_2} \tilde{I}(x,c)$ and $\bigvee_{x\in X_1} \tilde{I}(x,c) \leq \bigvee_{x\in X_2} \tilde{I}(x,c)$, then $\tilde{F}^{\pm}(X_1) \subseteq \tilde{F}^{\pm}(X_2)$ holds. In addition, for $\tilde{B}_1^{\pm} \subseteq \tilde{B}_2^{\pm}$, $H^{\pm}(\tilde{B}_1^{\pm}) = \{x \in G|\tilde{B}_1^{-}(c) \leq \tilde{I}(x,c) \leq \tilde{B}_1^{+}(c), \forall c \in M\}$ and $H^{\pm}(\tilde{B}_2^{\pm}) = \{x \in G|\tilde{B}_2^{-}(c) \leq \tilde{I}(x,c) \leq \tilde{B}_2^{+}(c), \forall c \in M\}$, then we have $H^{\pm}(\tilde{B}_1^{\pm}) \subseteq H^{\pm}(\tilde{B}_2^{\pm})$ since $\tilde{B}_1^{-}(c) \geq \tilde{B}_2^{-}(c)$ and $\tilde{B}_1^{+}(c) \leq \tilde{B}_2^{+}(c)$ for each $c \in M$.

(2) For any $x_a \in X$, there must be $\bigwedge_{x\in X} \tilde{I}(x,c) \leq \tilde{I}(x_a,c) \leq \bigvee_{x\in X} \tilde{I}(x,c)$ ($\forall c \in M$). Then we have $x_a \in H^{\pm}\tilde{F}^{\pm}(X)$. Thus, $X \subseteq H^{\pm}\tilde{F}^{\pm}(X)$. Also, for any $\tilde{B}_a^{\pm} \subseteq \tilde{F}^{\pm}H^{\pm}(\tilde{B}^{\pm})$, which is equal to $\tilde{B}_a^{\pm} \subseteq [\bigwedge_{x\in H^{\pm}(\tilde{B}^{\pm})} \tilde{I}(x,c), \bigvee_{x\in H^{\pm}(\tilde{B}^{\pm})} \tilde{I}(x,c)]$. Then we have $\tilde{B}_a^{-}(c) \geq \bigwedge_{x\in H^{\pm}(\tilde{B}^{\pm})} \tilde{I}(x,c)$ and $\tilde{B}_a^{+} \leq \bigvee_{x\in H^{\pm}(\tilde{B}^{\pm})} \tilde{I}(x,c)$ for any $c \in M$. Because $\bigwedge_{x\in H^{\pm}(\tilde{B}^{\pm})} \tilde{I}(x,c) \geq \tilde{B}^{-}(c)$ and $\bigvee_{x\in H^{\pm}(\tilde{B}^{\pm})} \tilde{I}(x,c) \leq \tilde{B}^{+}(c)$, we get $\tilde{B}_a^{-}(c) \geq \tilde{B}^{-}(c)$ and $\tilde{B}_a^{+}(c) \leq \tilde{B}^{+}(c)$ for any $c \in M$. Thus, $\tilde{B}^{\pm} \supseteq \tilde{F}^{\pm}H^{\pm}(\tilde{B}^{\pm})$ is obtained.

(3) According to (2), we know $X \subseteq H^{\pm}\tilde{F}^{\pm}(X)$. Then we have $\tilde{F}^{\pm}(X) \subseteq \tilde{F}^{\pm}H^{\pm}\tilde{F}^{\pm}(X)$ by (1). Since $\tilde{F}(X) = \tilde{B}^{\pm}$, and $\tilde{B}^{\pm} \supseteq \tilde{F}^{\pm}H^{\pm}(\tilde{B}^{\pm})$. Further, we obtain $\tilde{F}^{\pm}(X) \supseteq \tilde{F}^{\pm}H^{\pm}\tilde{F}^{\pm}(X)$. Meanwhile, $\tilde{F}^{\pm}(X) = \tilde{F}^{\pm}H^{\pm}\tilde{F}^{\pm}(X)$ holds. And it is easy to prove that $H^{\pm}(\tilde{B}^{\pm}) = H^{\pm}\tilde{F}^{\pm}H^{\pm}(\tilde{B}^{\pm})$ as above. □

The above mapping clearly satisfies the order-preserving Galois connection relation. In classical fuzzy concept-cognitive learning, the reverse order Galois link is convenient to represent the properties common to all objects. The sequence-preserving Galois link in this article shows that more objects have a larger range of properties. When considering multiple objects, the boundaries between them are better understood and exploring similarities among them becomes easier.

Property 2. Given a regular fuzzy-classical formal decision context $(G, M, \tilde{I}, D, J)$ and four mappings in Definition 2. For any object x, $(H^{-}\tilde{F}^{-}(x), \tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x))$ and $(H^{+}\tilde{F}^{+}(x), \tilde{F}^{-}H^{+}\tilde{F}^{+}(x) \cap \tilde{F}^{+}(x))$ are both interval-intent fuzzy concepts.

Proof. We only need to demonstrate (1) $\tilde{F}^{\pm}(H^{-}\tilde{F}^{-}(x)) = \tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x)$ and (2) $H^{\pm}(\tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x)) = H^{-}\tilde{F}^{-}(x)$. The same goes for another concept.

(1) Note that $\tilde{F}^{\pm}(H^{-}\tilde{F}^{-}(x)) = \tilde{F}^{-}(H^{-}\tilde{F}^{-}(x)) \cap \tilde{F}^{+}(H^{-}\tilde{F}^{-}(x))$ by Definition 3. According to Property 1, it is easy to obtain $\tilde{F}^{-}H^{-}\tilde{F}^{-}(x) = \tilde{F}^{-}(x)$. Thus, $\tilde{F}^{\pm}(H^{-}\tilde{F}^{-}(x)) = \tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x)$.

(2) According to Definition 2, we can get $\tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x) = [\tilde{I}(x,c), \bigvee_{x_i\in H^{-}\tilde{F}^{-}(x)} \tilde{I}(x_i,c)]_{\forall c}$. And $H^{-}\tilde{F}^{-}(x) = \{x_t \in G|\tilde{I}(x_t,c) \geq \tilde{I}(x,c), \forall c \in M\}$. Obviously, we have $H^{\pm}(\tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x)) = H^{\pm}([\tilde{I}(x,c), \bigvee_{x_i\in H^{-}\tilde{F}^{-}(x)} \tilde{I}(x_i,c)]_{\forall c}) = \{x_m \in G|\tilde{I}(x,c) \leq \tilde{I}(x_m,c) \leq \bigvee_{x_i\in H^{-}\tilde{F}^{-}(x)} \tilde{I}(x_i,c), \forall c \in M\} = \{x_t \in G|\tilde{I}(x_t,c) \geq \tilde{I}(x,c), \forall c \in M\} = H^{-}\tilde{F}^{-}(x)$.

In conclusion, there are $\tilde{F}^{\pm}(H^{-}\tilde{F}^{-}(x)) = \tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x)$ and $H^{\pm}(\tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x)) = H^{-}\tilde{F}^{-}(x)$, thus $(H^{-}\tilde{F}^{-}(x), \tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x))$ is an interval-intent fuzzy concept. □

Definition 4. Let $G/D = \{G^{d_k}|k = 1, 2, \cdots, |D|\}$ represent the partition of the object set based on their respective labels. For any object $x \in G^{d_k}$, $(H^{-}\tilde{F}^{-}(x), \tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x))$ and $(H^{+}\tilde{F}^{+}(x), \tilde{F}^{-}H^{+}\tilde{F}^{+}(x) \cap \tilde{F}^{+}(x))$ are upper-interval-intent fuzzy granular concept (short for UFGC and denote it by $(X^{+}, \tilde{B}^{\pm,u})$) and lower-interval-intent fuzzy granular concept (short for LFGC and denote it by $(X^{-}, \tilde{B}^{\pm,l})$), respectively. The interval-intent fuzzy granular concept spaces (under decision d_k) introduced by $UFGC$ and $LFGC$ are as follows:

$$GCS^{d_k,u} = \{(H^{-}\tilde{F}^{-}(x), \tilde{F}^{-}(x) \cap \tilde{F}^{+}H^{-}\tilde{F}^{-}(x))|x \in G^{d_k}\},$$

$$GCS^{d_k,l} = \{(H^{+}\tilde{F}^{+}(x), \tilde{F}^{-}H^{+}\tilde{F}^{+}(x) \cap \tilde{F}^{+}(x))|x \in G^{d_k}\}.$$

Obviously, UFGC and LFGC both are granular concepts derived from the same object whose attribute membership serves as the lower and upper bounds of the two interval-intent concepts, respectively. The concept space under each decision is combined into the overall concept space, which are defined as $GCS^{d,u} = \{GCS^{d_1,u}, GCS^{d_2,u}, \cdots, GCS^{d_{|D|},u}\}$ and $GCS^{d,l} = \{GCS^{d_1,l}, GCS^{d_2,l}, \cdots, GCS^{d_{|D|},l}\}$.

Algorithm 1: Construction of interval-intent fuzzy concept space

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Input: A quintuple (G, M, Ĩ, J, D).
Output: The interval-intent fuzzy granular concept
        spaces GCS^{d,u} and GCS^{d,l}.
According to the label, the object set G is divided and
  denoted as G/D = {G^{d_1}, G^{d_2}, ..., G^{d_{|D|}}};
for each G^{d_k} ∈ G/D do
    for each x_j ∈ G^{d_k} do
        Construct an UFGC (X^+, B̃^{±,u});
        GCS^{d_k,u} ← (X^+, B̃^{±,u});
        Construct a LFGC (X^-, B̃^{±,l});
        GCS^{d_k,l} ← (X^-, B̃^{±,l});
    end
    GCS^{d,u} ← GCS^{d_k,u}, GCS^{d,l} ← GCS^{d_k,l};
end
return GCS^{d,u} and GCS^{d,l}
```

Example 2. According to Definition 4, the four interval-intent fuzzy granular concept spaces in Table I can be derived as follows:

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$$\begin{aligned}GCS^{d_1,u} = \{&(\{x_1,x_3,x_6\},\{[0.32,0.48],[0.63,0.83]\}),\\&(\{x_2,x_3,x_5,x_6\},\{[0.36,0.48],[0.52,0.83]\}),\\&(\{x_3\},\{[0.48,0.48],[0.83,0.83]\}),\\&(\{x_3,x_4,x_5,x_6\},\{[0.38,0.48],[0.48,0.83]\}),\\&(\{x_3,x_5,x_6\},\{[0.4,0.48],[0.6,0.83]\}),\\&(\{x_3,x_6\},\{[0.41,0.48],[0.76,0.83]\})\}\end{aligned}$$

$$\begin{aligned}GCS^{d_1,l} = \{&(\{x_1\},\{[0.32,0.32],[0.63,0.63]\}),\\&(\{x_2\},\{[0.36,0.36],[0.52,0.52]\}),\\&(\{x_1,x_2,x_3,x_4,x_5,x_6\},\{[0.32,0.48],[0.48,0.83]\}),\\&(\{x_4\},\{[0.38,0.38],[0.48,0.48]\}),\\&(\{x_2,x_4,x_5\},\{[0.36,0.4],[0.48,0.6]\}),\\&(\{x_1,x_2,x_4,x_5,x_6\},\{[0.32,0.41],[0.48,0.76]\})\}\end{aligned}$$

$$\begin{aligned}GCS^{d_2,u} = \{&(\{x_7\},\{[0.8,0.8],[0.7,0.7]\}),\\&(\{x_8\},\{[0.81,0.81],[0.66,0.66]\}),\\&(\{x_7,x_8,x_9\},\{[0.77,0.81],[0.6,0.7]\}),\\&(\{x_{10}\},\{[0.91,0.91],[0.55,0.55]\}),\\&(\{x_7,x_8,x_9,x_{10},x_{11}\},\{[0.72,0.91],[0.45,0.7]\}),\\&(\{x_7,x_8,x_{12}\},\{[0.58,0.81],[0.61,0.7]\})\}\end{aligned}$$

$$\begin{aligned}GCS^{d_2,l} = \{&(\{x_7,x_9,x_{11},x_{12}\},\{[0.58,0.8],[0.45,0.7]\}),\\&(\{x_8,x_9,x_{11},x_{12}\},\{[0.58,0.81],[0.45,0.66]\}),\\&(\{x_9,x_{11}\},\{[0.72,0.77],[0.45,0.6]\}),\\&(\{x_{10},x_{11}\},\{[0.72,0.91],[0.45,0.55]\}),\\&(\{x_{11}\},\{[0.72,0.72],[0.45,0.45]\}),\\&(\{x_{12}\},\{[0.58,0.58],[0.61,0.61]\})\}\end{aligned}$$

B. Concept Re-cognition Process

Concept cognition is a process of starting from an object to find attributes and returning to find objects again. In the process of human cognition, there are often deviations or inaccuracies in the first cognition. Therefore, it is necessary to constantly update knowledge (concepts). The re-cognition process proposed in this paper can simulate the human thinking process to a certain extent, and find more accurate concept representation in complex and variable data.

Definition 5. Given the two interval-intent fuzzy concepts $(X_j^+,\tilde{B}_j^{\pm,u}) = (X_j^+,[\tilde{B}_j^{-,u},\tilde{B}_j^{+,u}]_{\forall c})$ and $(X_j^-,\tilde{B}_j^{\pm,l}) = (X_j^-,[\tilde{B}_j^{-,l},\tilde{B}_j^{+,l}]_{\forall c})$ induced by x_j, the definition of pseudo-intent in the re-cognition concept is as follows:

$$\tilde{B}_j^{\pm,p} = \{[\tilde{B}_j^{+,l}(c_i) - (1-er(x_j))\cdot(\tilde{B}_j^{+,l}(c_i) - \tilde{B}_j^{-,l}(c_i)),\\ \tilde{B}_j^{-,u}(c_i) + er(x_j)\cdot(\tilde{B}_j^{+,u}(c_i) - \tilde{B}_j^{-,u}(c_i))]|i=1,2,\cdots,|M|\}.$$

where $er(x_j) = \frac{|X_j^+|}{|X_j^+|+|X_j^-|}$.

The parameter $er(x_j)$ in the above definition is defined in terms of the extent ratio of UFGC and LFGC. Essentially to measure the magnitude of the fluctuation of the attribute value based on the number of objects in the extent. For example, if UFGC has a large number of extent, it means that for object x_j, there are more objects in the data set that are larger than its attribute value. Therefore, in the process of re-cognition, we give more tolerance to the larger attribute values, allowing the larger attribute values to be more retained.

The concept re-cognition process is divided into the following steps:

1. Find the two interval-intent fuzzy concepts induced by x_j, and compute the extent ratio $er(x_j)$.
2. Get the pseudo-intent $\tilde{B}_j^{\pm,p}$ of the re-cognition concept by Definition 5.
3. The pseudo-intent is used as the clue in the process of re-cognition, and the new concept $(H^{\pm}(B_j^{\pm,p}),F^{\pm}H^{\pm}(B_j^{\pm,p}))$ is obtained by Definition 3.
4. Repeat the above steps until every object in the space is re-recognized.

We call the concept newly learned from x_j the re-cognition concept, which denoted by $(X_j^r,\tilde{B}_j^{\pm,r}) = (H^{\pm}(B_j^{\pm,p}),F^{\pm}H^{\pm}(B_j^{\pm,p}))$. And newly obtained concept space a re-cognition concept space, denoted by RCS. According to Definition 3, it is easy to prove that the re-cognition concept is also an interval-intent fuzzy concept. At this time, the intent of the re-cognition concept is a subset of the pseudo-intent generated in the cognitive process, and the concept is more accurately depicted. We call other objects other than the primary original object in the extent of the re-cognition concept as approximate objects, and they have higher similarity with the original object.

Algorithm 2: Concept re-cognition process

Input: The set $G/D=\{G^{d_1},G^{d_2},\cdots,G^{d_{|D|}}\}$ of objects partitioned by labels, the set of extent ratio $ER=\{er(x_1),er(x_2),\cdots,er(x_{|G|})\}$, the concept spaces under each decision $GCS^{d,l}$ and $GCS^{d,u}$.

Output: Re-cognition concept space RCS.

```
for each G^{d_k} ∈ G/D do
    for each x_j ∈ G^{d_k} do
        Find the two interval-intent fuzzy concepts
          (X_j^+, B̃_j^{±,u}) = (X_j^+, [B̃_j^{-,u}, B̃_j^{+,u}]_{∀c}) and
          (X_j^-, B̃_j^{±,l}) = (X_j^-, [B̃_j^{-,l}, B̃_j^{+,l}]_{∀c}) induced
          by x_j in GCS^{d_k,u} and GCS^{d_k,l};
        B̃_j^{±,p} = {[B̃_j^{+,l}(c_i) − (1 − er(x_j) · (B̃_j^{+,l}(c_i) −
          B̃_j^{-,l}(c_i)), B̃_j^{-,u}(c_i) + er(x_j) · (B̃_j^{+,u}(c_i) −
          B̃_j^{-,u}(c_i))]|i = 1, 2, ··· , |M|};
        Get a re-cognition concept
          (X_j^r, B̃_j^{±,r}) = (H^±(B_j^{±,p}), F^±H^±(B_j^{±,p}))
          according to the mapping in Definition 3;
        RCS^{d_k} ← (X_j^r, B̃_j^{±,r})
    end
    RCS ← RCS^{d_k}
end
return RCS
```

Example 3. Object x_5 is taken as an example to illustrate the process of concept re-cognition. First, we obtain the ratio $er(x_5) = \frac{|\{x_3,x_5,x_6\}|}{|\{x_3,x_5,x_6\}|+|\{x_2,x_4,x_5\}|} = 0.5$ with the number of extent of the upper-interval-intent fuzzy granular concept and lower-interval-intent fuzzy granular concept. According to Definition 5, the pseudo-intent is $\tilde{B}_5^{\pm,p} = \{[0.38, 0.44], [0.54, 0.715]\}$. The re-cognition concept $(X_5^r, \tilde{B}_5^{\pm,r}) = (H^{\pm}(B_5^{\pm,p}), F^{\pm}H^{\pm}(B_5^{\pm,p})) = (\{x_5\}, \{[0.4, 0.4], [0.6, 0.6]\})$ is learned from the learning operator in Definition 3. After re-cognition of each object, we have the following re-cognition concept space:

$$
\begin{aligned}
RCS^{d_1} = \{&(\{x_1, x_6\}, \{[0.32, 0.41], [0.63, 0.76]\}),\\
&(\{x_2, x_5, x_6\}, \{[0.36, 0.41], [0.52, 0.76]\}),\\
&(\{x_3, x_5, x_6\}, \{[0.4, 0.48], [0.6, 0.83]\}),\\
&(\{x_4, x_5, x_6\}, \{[0.38, 0.41], [0.48, 0.76]\}),\\
&(\{x_5\}, \{[0.4, 0.4], [0.6, 0.6]\}),\\
&(\{x_5, x_6\}, \{[0.4, 0.41], [0.6, 0.76]\})\},
\end{aligned}
$$

$$
\begin{aligned}
RCS^{d_2} = \{&(\{x_7, x_9\}, \{[0.77, 0.8], [0.6, 0.7]\}),\\
&(\{x_8, x_9\}, \{[0.77, 0.81], [0.6, 0.66]\}),\\
&(\{x_9\}, \{[0.77, 0.77], [0.6, 0.6]\}),\\
&(\{x_{10}\}, \{[0.91, 0.91], [0.55, 0.55]\}),\\
&(\{x_9, x_{11}\}, \{[0.72, 0.77], [0.45, 0.6]\}),\\
&(\{x_{12}\}, \{[0.58, 0.58], [0.61, 0.61]\})\}.
\end{aligned}
$$

The process of concept re-cognition can be considered as the endeavor to identify objects based on their attributes, which essentially involves a rational expansion of the range of attribute values associated with a single object. In comparison to the preceding two interval granular concepts, re-cognition concept possesses the ability to mitigate the impact of noise to some extent. Simultaneously, it aligns with humans' second cognitive logic that individuals tend to associate with others who share similar characteristics.

C. Concept Clustering

In the previous section, we explored the re-cognition concepts which are typically numerous in quantity. Human memory is limited and can only retain the more significant concepts, necessitating compression and prioritization. During concept learning, a multitude of similar concepts with slight variations in attribute values may arise. However, precision at this level is not always essential; for instance, identifying a duck in a river without specifying its exact species suffices. In other words, people often treat such highly similar entities as belonging to the same concept. Building upon these notions, we introduce concept clustering into our proposed model.

Concept clustering is different from unsupervised clustering in machine learning. It aims to aggregate two or more supervised concepts into pseudo-concepts and reduce the size of concept space to improve classification efficiency. Concept clustering can be viewed as a compressed processing technique for labeled concepts.

1) Clustering Related Parameters:

Definition 6. Let $(X_i, \tilde{B}_i^{\pm})$ and $(X_j, \tilde{B}_j^{\pm})$ be two interval-intent fuzzy concepts, then the extent similarity is defined as follows:

$$\delta(X_i, X_j) = \frac{|X_i \bigcap X_j|}{|X_i \bigcup X_j|}. \tag{1}$$

Definition 7. Let $(X_i, \tilde{B}_i^{\pm})$ and $(X_j, \tilde{B}_j^{\pm})$ be two interval-intent fuzzy concepts, then the intent similarity is defined as follows:

$$\varphi(\tilde{B}_i^{\pm}, \tilde{B}_j^{\pm}) = 1 - \frac{\sum_c \left(\left|\tilde{B}_i^-(c) - \tilde{B}_j^-(c)\right| + \left|\tilde{B}_i^+(c) - \tilde{B}_j^+(c)\right|\right)}{2\,|M| + \sum_c \left(\left|\tilde{B}_i^+(c) - \tilde{B}_i^-(c)\right| + \left|\tilde{B}_j^+(c) - \tilde{B}_j^-(c)\right|\right)}, \tag{2}$$

where $|M|$ represents the number of attributes.

Property 3. For two concepts $(X_i, \tilde{B}_i^{\pm})$ and $(X_j, \tilde{B}_j^{\pm})$, we have:

(1) $0 \le \delta(X_i, X_j) \le 1$
(2) $0 \le \varphi(\tilde{B}_i^{\pm}, \tilde{B}_j^{\pm}) \le 1$

Proof. (1) It is immediate from Definition 6.

(2) Obviously, $0 \le \left|\tilde{B}_i^-(c) - \tilde{B}_j^-(c)\right| \le 1$, $0 \le \left|\tilde{B}_i^+(c) - \tilde{B}_j^+(c)\right| \le 1$. We get $0 \le \sum_c \left(\left|\tilde{B}_i^-(c) - \tilde{B}_j^-(c)\right| + \left|\tilde{B}_i^+(c) - \tilde{B}_j^+(c)\right|\right) \le 2 \cdot |M|$. Then we have $0 < \varphi(\tilde{B}_i^{\pm}, \tilde{B}_j^{\pm}) \le 1$. Notice that $\varphi(\tilde{B}_i^{\pm}, \tilde{B}_j^{\pm}) = 0$ if and only if for $\forall c \in M$, $\tilde{B}_i^+(c) = \tilde{B}_i^-(c) = 0$ and $\tilde{B}_j^+(c) = \tilde{B}_j^-(c) = 1$ (or the two sets of values are reversed). Thus, $0 \le \varphi(\tilde{B}_i^{\pm}, \tilde{B}_j^{\pm}) \le 1$. □

According to Definition 7, a higher degree of intent similarity indicates a greater level of similarity between two concepts. It is reasonable to assess this based on the distance between the boundaries of the concepts and the size of their respective intervals. A larger interval suggests that a concept has extensive intent coverage and is distinct from other concepts, thus indicating a high degree of similarity. However, existing concept-cognitive learning [34], [40] approaches only consider extent similarity (as defined in Definition 6) for clustering, overlooking the significance of attribute similarity. This approach fails to recognize that concepts with overlapping extents may not necessarily share similar intents. For instance, when making decisions about watermelons, large and small watermelons' concept extent are likely to intersect due to their inclusion relationship in terms of size. However, clustering them into the same concept would be misleading. Therefore, it is crucial to consider intent relationships when defining similarities.

2) Clustering Process: Based on the similarity of intent, this study integrates two or more concepts with high intent similarity to generate a novel concept. We refer to this new concept as a pseudo-concept, as its extent and intent are not strictly bound by the proposed mapping relationship. Within the concept space, we partition the blocks based on intent similarity. Concepts exhibiting an intent similarity exceeding the

Algorithm 3: Concept clustering

Input: Re-cognition concept space $RCS = \{RCS^{d_1}, RCS^{d_2}, \cdots, RCS^{d_{|D|}}\}$ and parameter γ.
Output: The pseudo-concept space $PC_\gamma = \{PC_\gamma^{d_1}, PC_\gamma^{d_2}, \cdots, PC_\gamma^{d_{|D|}}\}$.

```
for RCS^{d_k} ∈ RCS do
    PC_γ^{d_k} = ∅;
    for a concept (X_i^r, B̃_i^{±,r}) ∈ RCS^{d_k} do
        C_i^{d_k} = ∅;
        Calculate the intent similarity between each
          concept by Definition 7;
        Find the concept (X_j^r, B̃_j^{±,r}) ∈ RCS^{d_k} with
          the most intent similarity;
        if φ(B̃_i^{±,r}, B̃_j^{±,r}) > γ then
            C_i^{d_k} ← (X_j^r, B̃_j^{±,r})
        end
        if RCS^{d_k} = ∅ then
            End the clustering process under this label;
        end
        Cluster C_i^{d_k} as pseudo-concept (X_i^p, B̃_i^{±,p})
          according to definition 8;
        PC_γ^{d_k} ← (X_i^p, B̃_i^{±,p});
        Remove all the clustered concepts from
          RCS^{d_k};
    end
    PC_γ ← PC_γ^{d_k} ;
end
return PC_γ
```

threshold γ are grouped together and referred to as cluster C. It is evident that the re-cognition conceptual space comprises numerous clusters denoted by $RCS^{d_k} = \{C_1^{d_k}, C_2^{d_k}, \cdots, C_n^{d_k}\}$.

Definition 8. For a cluster origin $(X_{i_1}^r, \tilde{B}_{i_1}^{\pm,r})$ and all of its intent similarity in order from the largest to the smallest concept $(X_{i_2}^r, \tilde{B}_{i_2}^{\pm,r})$, $(X_{i_3}^r, \tilde{B}_{i_3}^{\pm,r})$, $\cdots$, $(X_{i_m}^r, \tilde{B}_{i_m}^{\pm,r}) \in C_i^{d_k}$, the extent and intent of the pseudo-concept $(X_i^p, \tilde{B}_i^{\pm,p})$ derived from the cluster $C_i^{d_k}$ are defined as follows:

$$X_i^p = X_{i_1}^r \cup X_{i_2}^r \cup \cdots \cup X_{i_m}^r,$$

$$\tilde{B}_i^{\pm,p}(c) = \frac{1}{2^{m-1}}(\tilde{B}_{i_1}^{\pm,r}(c) + \tilde{B}_{i_2}^{\pm,r}(c) + 2 \cdot \tilde{B}_{i_3}^{\pm,r}(c) + \cdots + 2^{m-2} \cdot \tilde{B}_{i_m}^{\pm,r}(c)), c \in M.$$

The pseudo-concept intent in definition 8 shows that the re-cognition concept that is clustered first will have more influence on the pseudo-concept. Therefore, we prioritize the participation of concepts with higher intent similarity in clustering, while concepts with lower intent similarity are involved later. Using pseudo-concepts can effectively reduce the size of the concept space. At the same time, the limitation of individual cognition is eliminated to a certain extent. Algorithm 3 shows the concrete process of concept clustering.

D. Object Classification Based on Intent Similarity

Since the new object to be classified will not be in the extent of any existing concept, the label judgment can only be made by the distance between the intent. Intent similarity in definition 7 can be seen as a distance measure. The object to be classified is regarded as an interval-intent fuzzy concept, whose interval-intent is composed of a single value of the membership degree of each attribute. Therefore, the formula in definition 7 is deformed to obtain the following distance definition:

Definition 9. Given a concept $(\{x_{new}\}, \tilde{B}_{new}^{\pm})$ to be classified derived from a new object x_{new}, its similarity distance with the pseudo-concept $(X^p, \tilde{B}^{\pm,p})$ is defined as follows:

$$S(x_{new}, X^p) = \frac{\sum_c \left(\left|\tilde{B}_{new}^{-}(c) - \tilde{B}_j^{-,p}(c)\right| + \left|\tilde{B}_{new}^{+}(c) - \tilde{B}_j^{+,p}(c)\right|\right)}{2 \cdot |M| + \sum_c \left(\left|\tilde{B}_j^{+,p}(c) - \tilde{B}_j^{-,p}(c)\right|\right)}, \quad (3)$$

where $|M|$ represents the number of attributes.

It can be seen that similarity distance and intent similarity are inversely proportional. The greater the intent similarity, the smaller the similarity distance. We obtain the pseudo-concept of the minimum similarity distance to determine the decision of the new object. The specific steps are represented in Algorithm 4.

Algorithm 4: New object label prediction

Input: The pseudo-concept space PC_γ and new object Δx in the test set.
Output: The predict label d_m of Δx.

```
H = ∅;
for each PC_γ^{d_k} ∈ PC_γ do
    for each (X^p, B̃^{±,p}) ∈ PC_γ^{d_k} do
        Compute S(Δx, X^p) according to fomula (3);
        H ← S(Δx, X^p)
    end
end
Get the minimum distance s = min(H) and
  corresponding minimum distance concept;
Find the label d_m for the minimum distance concept;
return Predict label d_m
```

E. Overall Procedure and Complexity Analysis

The overall flowchart of IFCRL is shown in Fig. 1. The model consists of three stages: 1) concept-cognitive process; 2) concept re-cognition process; 3) concept clustering and classification. For a fuzzy formal context with three decisions, the first concept learning is performed according to the mapping proposed in Definition 2, and two related concept spaces are obtained. Then, for each object, the extent ratio of their concepts in different concept spaces is calculated, so as to obtain the pseudo-intent used for re-cognition according to Definition 5. The pseudo-intent is learned to obtain the

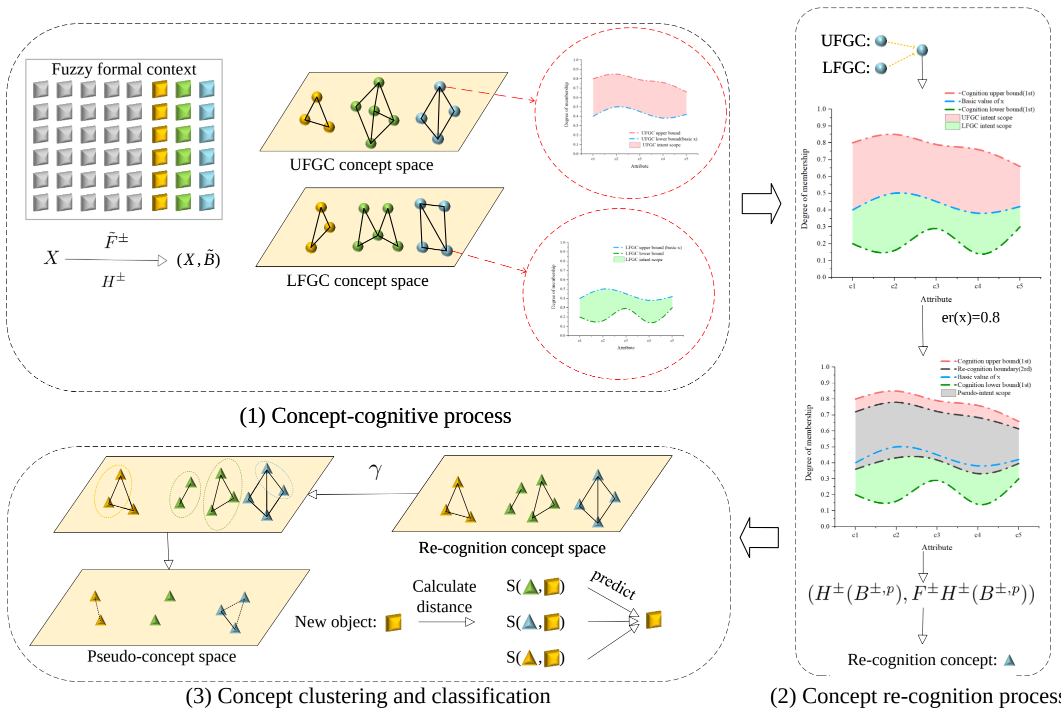


Fig. 1. General flow chart of IFCRL (1st and 2rd in the legend represent the concept-cognitive process and the concept re-cognition process, respectively).

re-cognition concept. Finally, the pseudo-concept space is obtained by concept clustering. According to Formula 3, the nearest concept to the newly added object is obtained to determine the label.

According to the aforementioned discussion, the time complexity of Algorithm 1 for constructing the initial concept space is $O(|G|^2|M|)$. The time complexity of re-cognition process is related to the number of objects. In the process of re-cognition of each object, it does not need to go through all the objects again, only need to search the objects in the extent union of UFGC and LFGC derived from the corresponding object. Assuming that the average number of objects in the extent union of two concepts (UFGC and LFGC) in two concept spaces is a, the time complexity of Algorithm 2 is $O(a|G||M|)$. In the process of clustering, the complexity of computing intent similarity is $O(|M|)$. Concepts that have already been clustered will not participate in the subsequent clustering process, so the number of concepts that need to be considered for clustering concepts will be gradually reduced each time. The choice of threshold γ is positively related to the complexity: a higher threshold will make the space more refined, thus increasing the total number of comparisons for intent similarity. The best case is that every concept in the space is clustered together at once, and the worst case is that no concept can be clustered with the others. Therefore, the time complexities of Algorithm 3 in the best and worst cases are $O(|G||M|)$, $O(|G|^2|M|)$, respectively. Obviously, when predicting the label, it is necessary to compare the new object with each pseudo-concepts. Therefore, Algorithm 4 takes $O(|PC_\gamma||M|)$. The overall complexity of IFCRL is $O(|G|^2|M|)$.

TABLE II
DETAILS OF THE EXPERIMENTAL DATASETS.

ID	Dataset	Object	Attribute	Class
1	Appendicitis	100	7	2
2	Parkinsons	197	22	2
3	Wine	198	33	3
4	Wpbc	200	33	2
5	Glass	214	9	6
6	Heart	270	13	2
7	Tic_tac_toe	958	9	2
8	Wine_red	1518	11	3
9	Segmentation	2310	19	7
10	Phoneme	5404	5	2
11	Mushroom	8124	22	2
12	Magic	19020	10	2

IV. EXPERIMENTS

In this section, we examine the rationality and effectiveness of the IFCRL algorithm proposed in a fuzzy context. We compare its classification accuracy with 14 algorithms and verify the effectiveness of the re-cognition method through ablation experiments. Additionally, we assess the importance of clustering by evaluating the rate at which space is compressed, and further explore how parameter γ affects clustering. The datasets utilized in this experiment are sourced from UCI and KEEL, with detailed information provided in Table II.

A. Experimental Setting

In order to adapt to the fuzzy environment in this paper, all datasets are normalized according to the following formula:

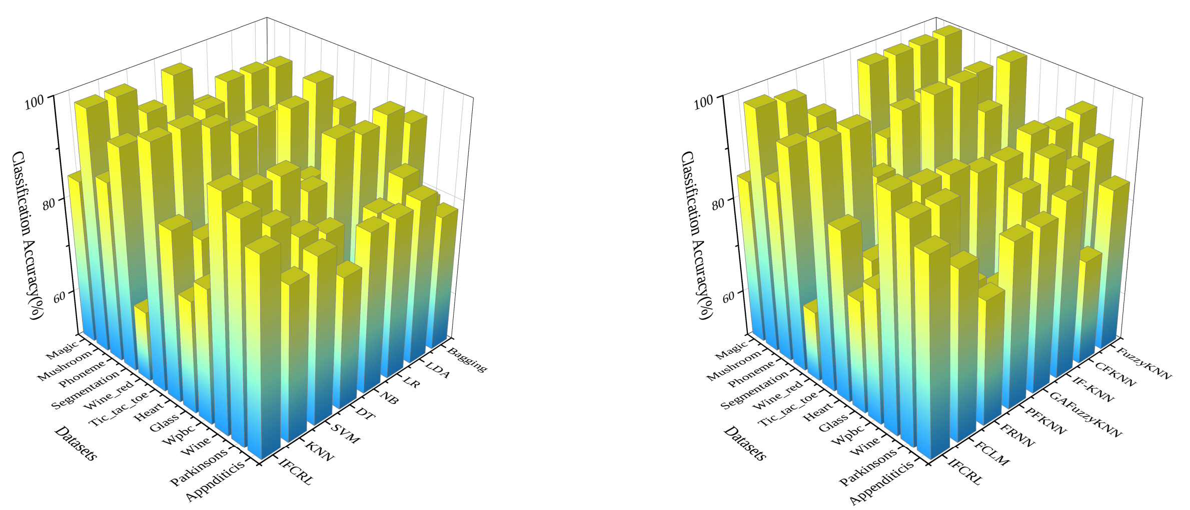
$$\tilde{I}(x_i, c_j) = \frac{v(x_i, c_j) - \min(v(c_j))}{\max(v(c_j)) - \min(v(c_j))} \quad (4)$$

TABLE III
COMPARISON OF ACCURACY (MEAN ± STANDARD DEVIATION%) AMONG IFCRL AND SEVEN CLASSIC CLASSIFICATION ALGORITHMS.

ID	γ	IFCRL	KNN	SVM	DT	NB	LR	LDA	Bagging
1	0.99	**90.50 ± 4.71**	82.00 ± 4.00	84.50 ± 5.22	77.50 ± 6.42	83.50 ± 5.02	83.50 ± 7.08	84.50 ± 6.50	78.50 ± 5.50
2	1.00	**94.87 ± 3.44**	90.77 ± 4.47	86.15 ± 2.61	84.10 ± 4.70	70.51 ± 4.62	82.82 ± 3.25	87.18 ± 3.24	79.74 ± 8.22
3	0.92	**97.94 ± 1.88**	95.58 ± 2.37	95.59 ± 3.01	90.29 ± 4.37	97.65 ± 2.88	96.18 ± 3.49	97.65 ± 2.88	93.82 ± 3.59
4	0.96	**78.00 ± 5.67**	75.25 ± 6.84	77.75 ± 7.20	69.75 ± 5.75	67.75 ± 6.56	**78.00 ± 7.42**	76.50 ± 6.91	64.75 ± 7.25
5	1.00	**73.49 ± 5.00**	66.98 ± 8.43	58.37 ± 7.16	66.51 ± 5.52	47.67 ± 8.27	54.88 ± 5.81	61.16 ± 3.46	24.42 ± 5.01
6	0.92	**85.37 ± 4.17**	80.83 ± 5.65	82.92 ± 4.82	72.08 ± 5.03	83.95 ± 5.67	82.50 ± 5.53	81.04 ± 3.11	72.41 ± 5.94
7	0.89	**100.00 ± 0.00**	**100.00 ± 0.00**	97.97 ± 0.59	94.64 ± 1.38	70.21 ± 1.85	95.10 ± 0.94	97.97 ± 0.59	90.83 ± 2.56
8	1.00	64.84 ± 3.52	61.87 ± 2.08	60.82 ± 2.66	**66.41 ± 2.75**	60.39 ± 2.64	62.43 ± 2.88	63.72 ± 2.39	52.11 ± 2.41
9	0.99	95.86 ± 1.19	94.55 ± 1.45	91.98 ± 1.42	**96.24 ± 0.77**	70.71 ± 1.60	90.45 ± 1.35	89.33 ± 1.59	85.79 ± 1.86
10	1.00	85.03 ± 0.45	**87.75 ± 1.24**	77.01 ± 1.28	87.20 ± 0.85	76.04 ± 1.39	75.04 ± 1.04	75.44 ± 0.71	72.53 ± 1.05
11	0.98	**100.00 ± 0.00**	99.98 ± 0.03	94.83 ± 1.13	**100.00 ± 0.00**	91.72 ± 0.31	94.70 ± 0.26	94.39 ± 0.34	93.67 ± 0.63
12	0.99	**84.34 ± 0.69**	82.92 ± 0.49	79.35 ± 0.57	81.31 ± 0.58	72.62 ± 0.52	79.07 ± 0.35	78.61 ± 0.56	77.83 ± 0.35
Average		**87.5200**	84.8733	82.2700	82.1692	74.3933	81.2225	82.2908	73.8667

TABLE IV
COMPARISON OF ACCURACY (MEAN ± STANDARD DEVIATION%) AMONG IFCRL AND SEVEN FUZZY-BASED CLASSIFICATION ALGORITHMS.

ID	γ	IFCRL	FCLM	FRNN	PFKNN	GAFuzzyKNN	IF-KNN	CFKNN	FuzzyKNN
1	0.99	**90.50 ± 4.71**	85.00 ± 6.12	76.00 ± 7.00	84.50 ± 4.15	84.50 ± 5.22	87.00 ± 5.57	72.00 ± 12.08	84.00 ± 7.68
2	1.00	**94.87 ± 3.44**	94.62 ± 3.53	76.67 ± 4.36	73.08 ± 6.90	89.23 ± 3.94	93.33 ± 2.05	88.72 ± 6.90	90.77 ± 5.15
3	0.92	**97.94 ± 1.88**	96.47 ± 2.20	95.88 ± 2.35	93.82 ± 3.59	93.53 ± 3.67	96.18 ± 3.96	94.71 ± 3.90	95.59 ± 2.71
4	0.96	**78.00 ± 5.65**	70.00 ± 6.80	73.00 ± 8.05	69.25 ± 6.62	72.75 ± 4.10	75.00 ± 7.83	56.50 ± 8.89	74.50 ± 7.57
5	1.00	**73.49 ± 5.00**	67.44 ± 3.82	53.26 ± 8.02	41.86 ± 5.60	61.40 ± 8.20	65.58 ± 7.77	62.79 ± 5.88	66.51 ± 4.67
6	0.92	**85.37 ± 4.17**	76.67 ± 3.81	80.83 ± 4.35	77.92 ± 4.08	79.58 ± 4.82	78.70 ± 4.92	70.00 ± 6.13	80.00 ± 6.19
7	0.89	**100.00 ± 0.00**	**100.00 ± 0.00**	66.61 ± 3.85	63.65 ± 2.04	99.84 ± 0.33	**100.00 ± 0.00**	92.29 ± 1.54	**100.00 ± 0.00**
8	1.00	**64.84 ± 3.52**	59.63 ± 1.55	56.63 ± 2.93	52.34 ± 2.33	61.15 ± 3.55	63.78 ± 2.53	56.18 ± 2.14	62.24 ± 4.44
9	0.99	**95.86 ± 1.19**	94.76 ± 1.28	82.86 ± 1.29	82.98 ± 0.92	93.98 ± 0.84	94.71 ± 0.77	92.57 ± 1.20	95.24 ± 0.72
10	1.00	85.03 ± 0.45	85.77 ± 1.01	70.61 ± 0.97	73.18 ± 1.41	86.93 ± 1.08	87.76 ± 1.12	82.54 ± 0.95	**88.27 ± 0.63**
11	0.98	**100.00 ± 0.00**	98.95 ± 1.02	94.09 ± 0.65	87.93 ± 0.48	**100.00 ± 0.00**	99.99 ± 0.02	99.95 ± 0.08	99.98 ± 0.03
12	0.99	**84.34 ± 0.69**	80.28 ± 0.88	64.97 ± 0.74	76.57 ± 1.05	83.04 ± 0.45	83.09 ± 0.75	76.23 ± 0.54	82.44 ± 0.41
Average		**87.5200**	84.1325	73.9508	73.0900	83.8275	85.4267	78.7067	84.9617



(a) Comparison with seven classical classification algorithms; (b) Comparison with seven fuzzy-based classification algorithms.

Fig. 2. Bar chart comparison of classification accuracy.

where $v(x_i, c_j)$ represents the value of x_i under attribute c_j in the original dataset, $\min(v(c_j))$ and $\max(v(c_j))$ represents, respectively, the minimum and maximum value of attribute c_j in the original data. The normalized value $\tilde{I}(x_i, c_j)$ is regarded as the fuzzy membership degree of object x_i under c_j.

In this paper, the experiment compares IFCRL with seven classical machine learning classification algorithms (KNN [43], SVM [44], DT [45], NB [46], LR [47], LDA [48], Bagging [49]) and seven fuzzy-based classification algorithms (FCLM [34], FRNN [50], PFKNN [51], GAFuzzyKNN [52], IF-KNN [53], CFKNN [54], FuzzyKNN [55]) where FCLM is a fuzzy concept-cognitive learning algorithm. 80 % and 20 % of each dataset serve as the training set and the test set, respectively. The dataset was executed 10 times with randomized data partitions and the results were averaged to assess each classification method. To ensure fairness, the aforementioned experiments were conducted using Python 3.10 on a personal computer equipped with an Intel(R) Core(TM) i5-10300H CPU @ 2.50GHz and 16 GB of memory.

B. Comparative Experimental Analysis

The accuracy of classic machine learning classification algorithms and the selected parameter γ by IFCRL are recorded in Table III. The average accuracy of IFCRL on 12 datasets is observed to be 87.52%. As depicted in Table III, it can be observed that LR performs well on one dataset, KNN performs well on two datasets, DT outperforms other algorithms on three datasets, while IFCRL demonstrates superior performance on nine datasets. Notably, IFCRL achieves the highest accuracy for both dataset 7 and dataset 11. Additionally, it exhibits lower standard deviation compared to other algorithms in four datasets(Wine, Tic_tac_toe, Phoneme, Mushroom).

Table IV shows the comparison of classification efficiency with fuzzy-based classification algorithm. The results show that IFCRL performs well on eleven datasets. It shows that we have the best classification performance and exhibits strong generalization capabilities among the selected fuzzy classifiers. Compared with FCLM algorithm based on fuzzy concept cognitive learning, IFCRL has better classification performance. The gap can be visualized more intuitively by creating a bar chart, as shown in Fig. 2.

C. Ablation Experiment

The proposed re-cognition process in this paper aims to acquire more precise knowledge. To validate the effectiveness of re-cognition, we initially learn UFGC and LGFC separately and conduct cognitive learning on them to obtain re-cognition concepts (RC). The aforementioned three types of concepts are individually tested through clustering experiments and classification accuracy assessments. In the experiment, clustering parameter γ was set as 1.00, 0.99, and 0.95 respectively for RC, UFGC, and LFGC which were each run ten times to calculate the average classification accuracy. The results are presented in Table V with the last column indicating the optimal accuracy score ratio for each round of experiments. And the experimental results demonstrate a strong resemblance between the classification effect of RC and UFGC on certain datasets (Wpbc, Glass, Mushroom). It can be observed that the final composite score ratio of these three concepts is 28:19:10. The overall impact of re-cognition is generally positive on the classification performance of most datasets, thus rendering the re-cognition process highly effective.

D. Theoretical Discussion and Parametric Analysis

In order to evaluate the clustering effect, the space compression rate is used to characterize the reduction degree of the spatial size after clustering, which is denoted by:

$$scr = \frac{|RCS| - |PC|}{|RCS|}, \tag{5}$$

where $|RCS|$ denotes the cardinality of concepts in the re-cognition space, and $|PC|$ denotes the cardinality of concepts in the clustered space. The purpose of concept clustering is to reduce the concept space. Fig. 3 shows the relationship between the space compression rate and parameter γ. Obviously, the relationship between the two is inversely proportional. The smaller the γ, the greater the space compression rate. For

TABLE V
RE-COGNITION LEARNING ABLATION EXPERIMENT.

ID	γ	RC	UFGC	LFGC	Score
	1.00	**91.50** ± **6.34**	90.00 ± 5.47	89.50 ± 4.71	
1	0.99	**90.00** ± **6.32**	87.50 ± 6.42	87.50 ± 7.15	3:0:0
	0.95	**89.00** ± **5.39**	88.00 ± 6.00	87.00 ± 6.00	
	1.00	94.61 ± 2.12	94.61 ± 2.12	**95.12** ± **1.38**	
2	0.99	92.30 ± 5.25	92.30 ± 5.25	**93.07** ± **5.38**	0:0:3
	0.95	87.69 ± 3.20	87.69 ± 3.20	**87.95** ± **3.98**	
	1.00	**95.88** ± **3.27**	95.58 ± 3.28	95.58 ± 3.28	
3	0.99	**97.06** ± **1.86**	**97.06** ± **1.86**	**97.06** ± **2.27**	3:2:1
	0.95	**96.17** ± **3.49**	**96.17** ± **3.49**	95.58 ± 3.17	
	1.00	**76.75** ± **4.75**	**76.75** ± **4.75**	72.00 ± 6.10	
4	0.99	**77.75** ± **7.02**	**77.75** ± **7.02**	69.00 ± 6.53	3:3:0
	0.95	**78.00** ± **5.67**	**78.00** ± **5.67**	66.49 ± 5.38	
	1.00	**69.30** ± **3.86**	**69.30** ± **3.86**	**69.30** ± **3.86**	
5	0.99	**71.16** ± **4.43**	**71.16** ± **4.43**	**71.16** ± **4.43**	3:3:3
	0.95	**58.83** ± **10.13**	**58.83** ± **10.13**	**58.83** ± **10.13**	
	1.00	**82.22** ± **4.98**	**82.22** ± **4.98**	80.92 ± 6.85	
6	0.99	78.89 ± 3.87	78.89 ± 3.87	**79.44** ± **2.22**	2:2:1
	0.95	**80.37** ± **5.69**	**80.37** ± **5.69**	80.19 ± 4.30	
	1.00	**100.00** ± **0.00**	**100.00** ± **0.00**	**100.00** ± **0.00**	
7	0.99	**100.00** ± **0.00**	**100.00** ± **0.00**	**100.00** ± **0.00**	3:2:2
	0.95	**100.00** ± **0.00**	99.90 ± 0.31	99.90 ± 0.31	
	1.00	**64.47** ± **1.35**	62.28 ± 1.35	61.95 ± 0.94	
8	0.99	**63.92** ± **0.86**	63.59 ± 1.78	60.20 ± 0.97	3:0:0
	0.95	**59.32** ± **0.94**	57.64 ± 0.82	57.01 ± 1.02	
	1.00	**95.44** ± **1.07**	94.20 ± 0.88	92.62 ± 0.70	
9	0.99	**96.35** ± **0.40**	96.11 ± 0.62	93.70 ± 0.96	3:0:0
	0.95	**88.25** ± **1.46**	87.86 ± 0.19	86.98 ± 1.48	
	1.00	84.21 ± 1.15	**88.47** ± **0.43**	77.06 ± 2.11	
10	0.99	82.91 ± 0.68	**87.29** ± **1.49**	75.67 ± 2.81	1:2:0
	0.95	**80.20** ± **0.63**	79.12 ± 1.20	74.56 ± 3.09	
	1.00	**100.00** ± **0.00**	**100.00** ± **0.00**	99.87 ± 0.23	
11	0.99	**100.00** ± **0.00**	**100.00** ± **0.00**	99.81 ± 0.25	3:3:0
	0.95	**100.00** ± **0.00**	**100.00** ± **0.00**	99.79 ± 0.30	
	1.00	83.51 ± 0.66	**83.86** ± **0.74**	80.39 ± 0.90	
12	0.99	**84.17** ± **0.61**	83.68 ± 0.60	80.76 ± 0.83	1:2:0
	0.95	83.29 ± 0.77	**83.45** ± **0.43**	80.11 ± 1.01	

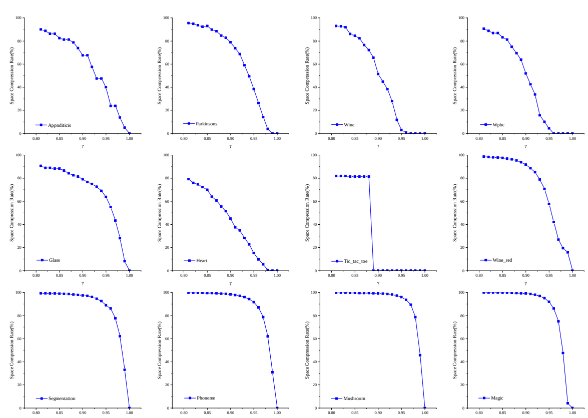


Fig. 3. The relationship between the space compression rate and the parameter γ on 12 datasets.

TABLE VI
RANKING OF THE FOURTEEN CLASSIFICATION ALGORITHMS.

ID	IFCRL	KNN	SVM	DT	NB	LR	LDA	Bagging	FCLM	FRNN	PFKNN	GAFuzzyKNN	IF-KNN	CFKN	FuzzyKNN
1	1.0	10.0	4.5	12.0	8.5	8.5	4.5	11.0	12.0	13.0	4.5	4.5	2.0	14.0	7.0
2	1.0	4.5	9.0	10.0	15.0	11.0	8.0	12.0	2.0	13.0	14.0	6.0	3.0	7.0	4.5
3	1.0	10.0	8.5	15.0	2.5	5.5	2.5	12.5	4.0	7.0	12.5	14.0	5.5	11.0	8.5
4	1.5	5.0	3.0	10.0	12.0	1.5	4.0	13.0	10.0	8.0	11.0	9.0	6.0	14.0	7.0
5	1.0	2.0	9.0	3.5	12.0	10.0	8.0	14.0	2.0	11.0	13.0	7.0	5.0	6.0	3.5
6	1.0	6.5	3.0	13.0	2.0	4.0	5.0	12.0	12.0	6.5	11.0	9.0	10.0	14.0	8.0
7	3.0	3.0	7.5	10.0	13.0	9.0	7.5	12.0	3.0	14.0	15.0	6.0	3.0	11.0	3.0
8	2.0	7.0	9.0	1.0	10.0	5.0	4.0	14.0	11.0	12.0	13.0	8.0	3.0	11.0	6.0
9	2.0	6.0	9.0	1.0	15.0	10.0	11.0	12.0	4.0	14.0	13.0	7.0	5.0	8.0	3.0
10	6.0	3.0	8.0	4.0	9.0	11.0	10.0	13.0	6.0	14.0	12.0	5.0	2.0	7.0	1.0
11	2.0	5.5	8.0	2.0	13.0	9.0	10.0	12.0	8.0	11.0	14.0	2.0	4.0	7.0	5.5
12	1.0	4.0	7.0	6.0	13.0	8.0	9.0	10.0	7.0	14.0	11.0	3.0	2.0	12.0	5.0
Average	1.96	4.96	8.00	7.00	11.21	8.13	7.71	12.42	6.75	10.96	13.29	6.88	4.58	10.83	5.33

datasets with a small number of objects, the space compression rate decreases evenly with the increase of γ. For datasets with a large number of objects, the image drops off a cliff when $\gamma \in [0.95, 1]$. The image of dataset 7 plummets around $\gamma = 0.88$, indicating that the intent similarity between concepts is relatively concentrated. Therefore, concept clustering is reasonable and efficient.

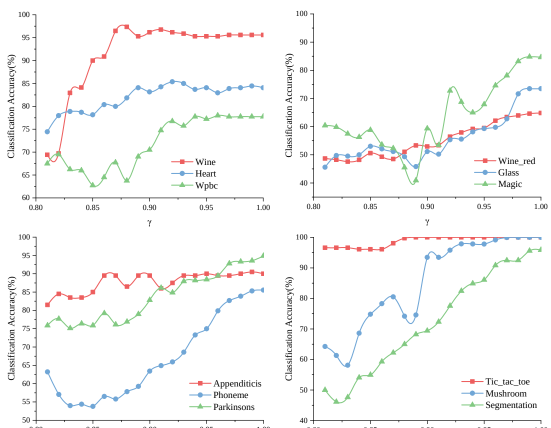


Fig. 4. The relationship between the parameter γ and classification accuracy of IFCRL on 12 datasets.

Combined with algorithm 3, we know that the classification accuracy of IFCRL is closely related to parameter γ. Therefore, it is necessary to discuss the effect of parameter γ on classification accuracy. For different datasets, it is necessary to choose the appropriate threshold γ in order to achieve the optimal accuracy. According to Fig. 3, when $\gamma = 0.81$, the compression ratio in space is all higher than 80%, and some even reach 95%. Thus the discussion of the parameter γ is in the range [0.81,1]. We set the step size to 0.01 and test the parameter within the range [0.81,1], that is, $\gamma \in \{0.81, 0.82, \cdots, 1\}$. And for the same parameter, 10 experiments were conducted and the average accuracy was calculated. The specific change trend is shown in Fig. 4. We can observe that most classification accuracy presents a gradual upward trend, and individual data sets have a state of first increase and then decrease. The validity of clustering is substantiated to a certain extent.

E. Statistical Significance Analysis

The Friedman test is a rank-based statistical method used to determine whether there are significant differences in the average performance of multiple models across multiple datasets. The accuracy of all classification algorithms on the 12 datasets is ranked and represented in Table VI. The Friedman statistic is calculated as follows:

$$F_F = \frac{(N-1)\chi_F^2}{N(k-1)-\chi_F^2} \sim F(k-1,(k-1)(N-1)), \quad (6)$$

where $\chi_F^2 = \frac{12N}{k(k+1)}(\sum_{i=1}^{k} R_i^2 - \frac{k(k+1)^2}{4})$, k and N are the number of different algorithms and datasets, respectively. $R_i = \frac{1}{N}\sum_{j=1}^{N} r_j^i$ indicates the average rank of i-th algorithm on all the datasets, and r_j^i indicates the rank of i-th algorithm on j-th dataset. We assume that there is no significant difference between all algorithms, and if $F_F > F(k-1,(k-1)(N-1))$ then reject the null hypothesis. According to the number of datasets and the number of algorithms in this paper, $\chi_F^2 = 68.93$ is obtained. We plug this into formula (6) to obtain $F_F = 8.71 > 1.79 = F(13, 143)$ in $\alpha = 0.05$. Therefore, the null hypothesis does not hold and there is a significant difference between all algorithms.

The Friedman test can only be used to determine whether there is a significant difference between the measurements of multiple models, but it cannot know whether there is a difference between any two models, which is what the Nimenyi test aims to solve. The critical range CD formula for the difference between the average order values is calculated from the Nemenyi test as follows:

$$CD = q_\alpha \sqrt{\frac{k(k+1)}{6N}}, \quad (7)$$

where q_α is the critical value corresponding to the significance level, k and N are the number of different algorithms and datasets, respectively. In this paper, q_α is determined to be 3.383 based on the degrees of freedom and significance level of the test. We plot the average ranking and critical difference (CD) of the algorithms in Fig. 5. The results indicate that when $\alpha = 0.05$, there are statistically significant distinctions between IFCRL and CFKNN as well as the other four classification methods.

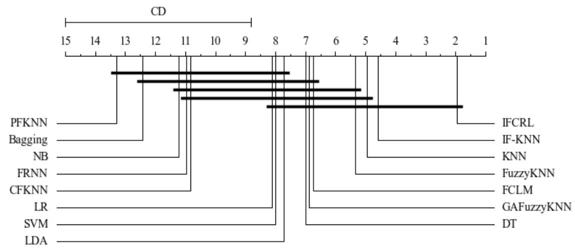


Fig. 5. CD plot of all classification methods used in the experiment.

V. Conclusion

In this paper, we propose a novel interval-intent fuzzy concept re-cognition learning model called IFCRL. Firstly, we provide the definition of an interval-intent fuzzy concept, where the intent of classical fuzzy concepts is transformed from single-valued to interval-valued form. Based on this, we introduce a concept re-cognition process that keeps the value of the property within a reasonable range. This process serves as a secondary learning mechanism on the data and effectively mitigates the impact of extreme value noise. Furthermore, the introduction of concept clustering based on intent similarity not only simplifies the concept space but also overcomes the cognitive limitations encountered in previous clustering processes. Finally, extensive experiments are conducted on 12 datasets from UCI and KEEL databases to demonstrate our method's advanced classification ability.

This work presents a novel perspective on concept learning, expanding the boundaries of cognitive understanding beyond unidirectional limitations and providing more precise descriptions of cognitive outcomes. The issue of the high similarity between the RC and UFGC, however, still needs to be addressed. It is imperative to explore a more rational re-cognition process in order to tackle the problem of ineffective re-cognition. And it is important to acknowledge that human cognition is an intricate process, which goes beyond simple set intersection operations when considering secondary or multiple cognitions. Additionally, there are deeper issues such as conceptual reasoning, associative memory, selective forgetting, etc., that warrant further investigation in future research.

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