Fuzzy-Granular Concept-Cognitive Learning via Three-Way Decision: Performance Evaluation on Dynamic Knowledge Discovery

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***Abstract*—Concept-cognitive learning (CCL) and three-way decision (3WD) models provide powerful techniques for knowledge discovery. Some early attempts in the field have successfully combined CCL and 3WD, i.e., three-way concept learning. However, only a few attempts were made to combine CCL with 3WD in a dynamic fuzzy context due to two challenges: 1) Three-way CCL incapability; 2) The current incremental three-way concept learning mechanism is insufficient to model real-time updating cognitive procedure. Hence, this article first shows some new standpoints on improving fuzzy-based CCL accuracy and then proposes fuzzy-granular three-way concept-cognitive learning (F3WG-CCL) for concept modeling and dynamic knowledge learning. Specifically, we first define a new F3WG-concept to characterize the knowledge embedded in fuzzy data. Furthermore, a big concept priority principle and an update mechanism are borrowed for concept recognition and dynamic concept cognition. Finally, we show that F3WG-CCL can be implemented simultaneously via theoretical guarantee and sufficient experimental, including 1) achieving state-of-the-art dynamic knowledge learning; 2) demonstrating that the three-way concept is effective in a fuzzy context; and 3) discovering that the big concept is valuable for fuzzy concept recognition. Our work will provide a powerful approach to research fuzzy-based CCL and dynamic knowledge discovery.**

***Index Terms*—Concept-cognitive learning (CCL), fuzzy context, granular computing (GrC), knowledge discovery, three-way decision (3WD).**

I. Introduction

UNCERTAINTY is a universal phenomenon in nature [1]. The description and thinking of the uncertainty phenomenon have become an active topic in many fields, such as philosophy, cognitive science, computer science, artificial intelligence, and many more. With the rapid development of artificial intelligence, various uncertainty information processing methods have attracted significant attention recently. In order to make the machine have the intelligence of human brain in perceiving, reasoning, and making decision processes, all of these require an in-depth study concerning uncertain information processing. A central notion of this processing is to represent and discover knowledge embedded in data.

As we all know, granular computing (GrC) is an interesting and influential theory for studying uncertainty knowledge discovery and has been shown to be a promising information processing paradigm [2]. Currently, various theoretical models of GrC dealing with uncertainty in knowledge appear basically, including approximate operator in rough set, fuzzy membership of fuzzy set, decision rules for three-way decision (3WD), fuzzy intent of fuzzy concept, etc. Moreover, numerous viewpoints of GrC have been successfully applied to meet different requirements of knowledge discovery [3], [4]. Among them, fuzzy-based concept-cognitive learning (CCL), as a newly emerging theory, is a straightforward approach to conveniently and effectively depict the general and objective essence of things ontology [5], [6].

CCL is the science of cognition and learning things via concepts [7]. In the intelligence era of data and knowledge dual-driven, the CCL theory and method is popular and hot in artificial intelligence and cognitive computing due to its ability to represent, learn, and cognitive concepts from data. In some sense, CCL is a novel and practical theoretical framework for a human to explore the cognitive mechanism of the brain [8], [9]. Now, inspired by machine learning and cognitive computation, CCL theory and method have been explored from different aspects, such as granular concept [10], fuzzy concept [11], three-way concept [12], fuzzy three-way concept [13], weighted fuzzy concept [14], concept tree [15], etc. Meanwhile, numerous concept learning ways, such as progressive learning [6], incremental learning [16], and semisupervised learning [17], have been presented to meet different requirements of concept learning.

In addition, for different problem scenarios, CCL also puts forward its unique views and solutions from its perspective, such as a multi-attention CCL for concept generation on the handwritten numeral [18], multilevel cognitive concept learning to recognize and distinguish micro-expressions [19], multiview concept learning for data representation [20], fuzzy-based CCL to tumor diagnosis analysis [7], two-way CCL with multisource

Manuscript received 22 August 2023; accepted 12 October 2023. Date of publication 19 October 2023; date of current version 1 March 2024. This work was supported by the National Natural Science Foundation of China under Grant 61976245 and Grant 61976120. Recommended by Associate Editor C. Mencar. *(Corresponding author: Weihua Xu.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TFUZZ.2023.3325952.

Digital Object Identifier 10.1109/TFUZZ.2023.3325952

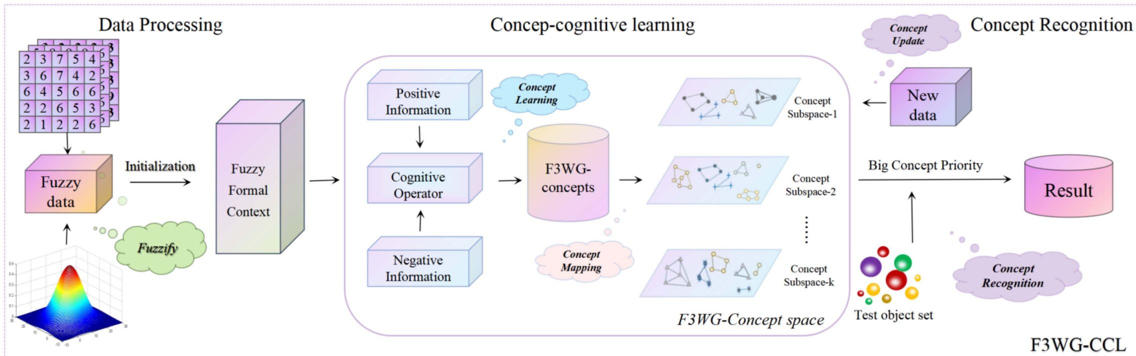


Fig. 1. Framework of the proposed method.

formal context [21], memory-based CCL for dynamic fuzzy data classification and knowledge fusion [9], and many more. A detailed analysis of various concept models under the CCL theory can be found in [22], and the detailed developmental stages of CCL from a GrC perspective can be found in [6]. All in all, CCL has made many meaningful achievements, but there are still many problems to be solved and further explored in this field. Therefore, enhancing and enriching the research in this area of CCL via various theories, frameworks, models, and viewpoints is necessary.

It is worth mentioning that the fuzzy-based CCL is another critical representative study of CCL for discovering the knowledge structure and information embedded in fuzzy data. The authors in [11] and [14] propose CCL based on the fuzzy or weighted fuzzy concept for the concept classification. In addition, the authors in [7], [9], and [13] discuss fuzzy three-way concepts grounded on different information granularities for concept learning. Note that these studies unify the view of using pseudoconcept for concept recognition. Moreover, the existing fuzzy-based CCL also has some ambiguous views when dealing with a classification task, mainly reflected in the following.

1) Do the fuzzy three-way concepts have advantages over regular fuzzy concepts?
2) How does the initial information granular influence fuzzy three-way concept learning?
3) What is the mechanism by which pseudoconcepts affect classification performance?

Based on the above analysis, we first show, in this article, some new standpoints on the improvement of CCL accuracy under a fuzzy environment.

1) Fuzzy three-way concepts have superiority over regular fuzzy concepts in the integrity of knowledge depiction and reduce cognitive bias (i.e., the fuzzy three-way concept is effective).
2) The fuzzy-granular three-way concept for CCL (F3WG-CCL) is more efficient and significantly does not depend on the initial information granular (i.e., the fuzzy-granular CCL is adequate);
3) Big concept priority principle in a fuzzy environment (i.e., the big concepts are more valuable).

The big concept refers to a formal concept with a larger extent in concept space, and a more detailed explanation can be found in Section IV-C. Alternatively, the current incremental three-way concept learning mechanism study cannot model real-time updating cognitive procedure. Following these thoughts, we build F3WG-CCL concept and performance evaluation on dynamic knowledge discovery. The framework of the proposed method is shown in Fig. 1. The main aspects of contributions are as follows.

1) We discover some ambiguous views of existing fuzzy-based CCL when dealing with the classification task. Then, we first present some new standpoints on improving CCL accuracy under a fuzzy environment, and the performance of the proposed standpoints is experimentally justified by conducting experiments.
2) We define a new F3WG-concept for CCL, which can effectively reduce the complexity of fuzzy three-way concept learning.
3) We design an update mechanism of cognitive operator for dynamic data updating and take advantage of past knowledge to achieve effective dynamic concept learning.
4) We propose a concept recognition mechanism based on the big concept priority principle for fuzzy-based CCL, which can enhance the classification performance of fuzzy data and help explain the critical reason for the validity of the pseudoconcept.

The rest of this article is organized as follows. We revisit several notions of fuzzy-based CCL in Section II. Section III defines a new fuzzy granular concept via three-way concept analysis. Furthermore, the proposed F3WG-CCL method is shown in Section IV. The experimental analysis is presented in Section V. Finally, Section VI concludes this article.

II. Preliminaries

This section revisits several notions of fuzzy-based CCL, then gives several forms of fuzzy concept.

A. Fuzzy Decision Formal Context

In this section, we characterize a fuzzy decision formal context and then introduce several fuzzy concept forms in CCL. These notions also can be found in [6], [10], [11], and [23].

Let $(\Omega, \Psi, \widetilde{I})$ and (Ω, D, J) be two formal contexts, then a quintuple $(\Omega, \Psi, \widetilde{I}, D, J)$ is referred to as a fuzzy decision formal context, where the following holds.

1) $\Omega = \{x_1, x_2, \ldots, x_n\}$ is a nonempty finite object set.
2) $\Psi = \{a_1, a_2, \ldots, a_m\}$ is a nonempty finite attribute set.
3) $\widetilde{I} = \{< (x,a), \widetilde{I}(x,a) > |(x,a) \in \Omega \times \Psi\}$ is a fuzzy relation on $\Omega \times \Psi$, where $\widetilde{I} : \Omega \times \Psi \to [0,1]$, $\widetilde{I}(x,a)$ denotes the membership degree of x with respect to a.
4) $\Omega/D = \{D_1, D_2, \ldots, D_l\}$ is a decision division based on decison class D, where $D = D_1 \cup D_2 \cup, \ldots, \cup D_l$.
5) $J : \Omega \times D \to \{D_1, D_2, \ldots, D_l\}$ is a binary relation on $\Omega \times D$, where $J : \Omega \times D \to \{0,1\}$.

In fuzzy decision formal context $(\Omega, \Psi, \widetilde{I}, D, J)$, 2^Ω, and 2^D are the power sets of Ω and Ψ, Γ^Ψ is the union of all fuzzy sets on Ψ. A pair of set-valued mappings $\widetilde{\mathcal{L}} : 2^\Omega \to \Gamma^\Psi$ and $\mathcal{H} : \Gamma^\Psi \to 2^\Omega$ can be considered as two cognitive operators.

Furthermore, for any $X_1, X_2 \subseteq \Omega$ and $\widetilde{A} \in \Gamma^\Psi$, the following statements hold:

1) $X_1 \subseteq X_2 \Rightarrow \widetilde{\mathcal{L}}(X_2) \subseteq \widetilde{\mathcal{L}}(X_1)$;
2) $\widetilde{\mathcal{L}}(X_1 \cup X_2) \subseteq \widetilde{\mathcal{L}}(X_2) \cap \widetilde{\mathcal{L}}(X_1)$;
3) $\mathcal{H}(\widetilde{A}) = \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}(x)\}$.

Thus, given a fuzzy decision formal context $(\Omega, \Psi, \widetilde{I}, D, J)$, a pair $(X, \widetilde{A})$ is called a fuzzy concept if only $\widetilde{\mathcal{L}}(X) = \widetilde{A}$ and $\mathcal{H}(\widetilde{A}) = X$ hold, and then X and $\widetilde{A}$ are the extent and intent of the fuzzy concept $(X, \widetilde{A})$. From a view of philosophy, Section II-A can characterize three clear cognitive semantic interpretations. Then, using these two set-valued mappings (i.e., $\widetilde{\mathcal{L}}$ and $\mathcal{H}$), we can discover the knowledge via a cognitive concept from the formal context.

B. Several Forms of Fuzzy Concept

During the past few years, CCL has been an active research topic in knowledge discovery, especially fuzzy-based CCL is a straightforward, simple, and efficient method to discover knowledge embedded in continuous data. This section mainly introduces some popular fuzzy concept forms.

Definition 1: A quintuple $(\Omega, \Psi, \widetilde{I}, D, J)$ is a fuzzy decision formal context, for any $X \subseteq \Omega$, $A \subseteq \Psi$, and $\widetilde{A} \in \Gamma^\Psi$, the cognitive operator $\widetilde{\mathcal{L}}$ and $\mathcal{H}$ can be defined as follows:

$$\widetilde{\mathcal{L}}(X)(a) = \bigwedge_{x \in X} \widetilde{I}(x,a), a \in \Psi$$

$$\mathcal{H}(\widetilde{A}) = \{x \in \Omega | \quad \forall a \in A, \widetilde{A}(a) \leqslant \widetilde{I}(x,a)\}.$$

The 3WD based on symbols-meaning-value (SMV) conceptual model, coined by Yao [24] in his seminal article, is a compelling and interesting three-level GrC paradigm for studying the high-level conception of data science. Inspired by this theory, the authors in [25] and [26] combine 3WD with the formal concept to study formal concept analysis from positive and negative information, i.e., three-way formal concept analysis. Subsequently, CCL models based on 3WD have been investigated in different fields to address different problem needs [7], [27].

Let $\widetilde{I}^- = \{< (x,a), 1 - \widetilde{I}(x,a) > |(x,a) \in \Omega \times \Psi\}$ be the complement of $\widetilde{I}$, where $1 - \widetilde{I}(x,a)$ reflects the nonmembership degree. Then, we denote by $\widetilde{I}^-(x,a) = 1 - \bar{I}(x,a)$. Assumed that the $\widetilde{\mathcal{L}}^- : 2^\Omega \to \Gamma^\Psi$ and $\mathcal{H}^- : \Gamma^\Psi \to 2^\Omega$ are a pair of negative cognitive operators. In this case, we say that $\widetilde{\mathcal{L}}$ and $\mathcal{H}$ are a pair of positive cognitive operators.

Definition 2: Given a fuzzy decision formal context $(\Omega, \Psi, \widetilde{I}, D, J)$, for any $X \subseteq \Omega$ and $\widetilde{A} \in \Gamma^\Psi$, the negative cognitive operator $\widetilde{\mathcal{L}}^-$ and $\mathcal{H}^-$ can be defined as follows:

$$\widetilde{\mathcal{L}}^-(X)(a) = \bigwedge_{x \in X} \widetilde{I}^-(x,a), a \in \Psi$$

$$\mathcal{H}^-(\widetilde{A}) = \{x \in \Omega | \quad \forall a \in A, \widetilde{A}(a) \leqslant \widetilde{I}^-(x,a)\}$$

where $\widetilde{A}$ is the fuzzy set on the complement of Ψ.

Then, to simultaneously express positive and negative information, we can combine the positive and negative cognitive operators to form a unique cognitive operator called the three-way cognitive operator.

Definition 3: A quintuple $(\Omega, \Psi, \widetilde{I}, D, J)$ is a fuzzy decision formal context. For any $X \subseteq \Omega$ and $\widetilde{A_1}, \widetilde{A_2} \in \Gamma^\Psi \times \Gamma^\Psi$, the fuzzy three-way concept cognitive operator $\widetilde{\mathcal{L}}^\nabla : 2^\Omega \to \Gamma^\Psi \times \Gamma^\Psi$ and $\mathcal{H}^\nabla : \Gamma^\Psi \times \Gamma^\Psi \to 2^\Omega$ are defined by

$$\widetilde{\mathcal{L}}^\nabla(X) = (\widetilde{\mathcal{L}}(X), \widetilde{\mathcal{L}}^-(X))$$

$$\mathcal{H}^\nabla(\widetilde{A_1}, \widetilde{A_2}) = \mathcal{H}(\widetilde{A_1}) \cap \mathcal{H}^-(\widetilde{A_2}).$$

Then, we say that $(X, (\widetilde{A}_1, \widetilde{A}_2))$ is a fuzzy three-way concept, if $\widetilde{\mathcal{L}}^\nabla(X) = (\widetilde{A}_1, \widetilde{A}_2), \mathcal{H}^\nabla(\widetilde{A}_1, \widetilde{A}_2) = X$ hold. Obviously, $(\mathcal{H}^\nabla \widetilde{\mathcal{L}}^\nabla(X), \widetilde{\mathcal{L}}^\nabla(X))$ and $(\mathcal{H}^\nabla(\widetilde{A}_1, \widetilde{A}_2), \widetilde{\mathcal{L}}^\nabla \mathcal{H}^\nabla(\widetilde{A}_1, \widetilde{A}_2))$ represent the object-oriented and attribute-oriented fuzzy three-way concept, respectively. Moreover, $(X, (\widetilde{A}_1, \widetilde{A}_2))$ is referred to as a subconcept of $(X', (\widetilde{A}'_1, \widetilde{A}'_2))$ if $X \subseteq X'$ or $(\widetilde{A}'_1, \widetilde{A}'_2) \geq (\widetilde{A}_1, \widetilde{A}_2)$, denoted by $(X, (\widetilde{A}_1, \widetilde{A}_2)) \leq (X', (\widetilde{A}'_1, \widetilde{A}'_2))$.

According to Definitions 1–3, some scholars have constructed different fuzzy concept forms (e.g., $(\mathcal{H}\widetilde{\mathcal{L}}(X), \widetilde{\mathcal{L}}(X))$, $(H(\widetilde{A}), \widetilde{\mathcal{L}}(\widetilde{A}))$, $(H\widetilde{\mathcal{L}}(X), \widetilde{\mathcal{L}}(X), w)$, $(\mathcal{H}^\nabla \widetilde{\mathcal{L}}^\nabla(X), \widetilde{\mathcal{L}}^\nabla(X))$, and $(\mathcal{H}^\nabla(\widetilde{A}_1, \widetilde{A}_2), \widetilde{\mathcal{L}}^\nabla \mathcal{H}^\nabla(\widetilde{A}_1, \widetilde{A}_2)))$ for CCL and achieved good results in data classification. More details about them can be found in the corresponding papers [7], [10], [11], [13], [14]. Nevertheless, we can also know that learning the attribute-oriented concept in a fuzzy context is sometimes immensely challenging without giving the initial clues $\widetilde{A}_1, \widetilde{A}_2 \in \Gamma^\Psi \times \Gamma^\Psi$. The object-oriented concept is usually utilized in fuzzy concept learning to complete a particular learning task, such as classification or recognition. Hence, hereinafter we only discuss the CCL situation under the object-oriented fuzzy three-way concept.

III. Fuzzy Granular Concept Via Three-Way Concept Analysis

In this section, we aim to introduce a novel fuzzy concept based on the three-way concept analysis (i.e., F3WG-concept).

We will first show some new notions and properties for the proposed concept as follows.

Definition 4: A quintuple $(\Omega, \Psi, \widetilde{I}, D, J)$ is a fuzzy decision formal context, for any $X \subseteq \Omega$ and $\widetilde{A} \in \Gamma^{\Psi}$, the positive cognitive operators $\widetilde{\mathcal{L}}$ and $\mathcal{H}$ can be defined as follows:

$$\widetilde{\mathcal{L}}(X)(a) = \bigwedge_{x \in X} (\widetilde{I}(x,a) - \mu(a)), a \in \Psi$$

$$\mathcal{H}(\widetilde{A}) = \{x \in \Omega | \ \forall a \in A, \widetilde{I}(x,a) \geq \widetilde{A}(a) - \mu(a)\}.$$

Similarly, the negative cognitive operators $\widetilde{\mathcal{L}}^{-}$ and $\mathcal{H}^{-}$ can be defined as follows:

$$\widetilde{\mathcal{L}}^{-}(X)(a) = \bigwedge_{x \in X} (\widetilde{I}^{-}(x,a) - v(a)), a \in \Psi$$

$$\mathcal{H}^{-}(\widetilde{A}) = \{x \in \Omega | \ \forall a \in A, \widetilde{I}^{-}(x,a) \geq \widetilde{A}(a) - v(a)\}$$

where $0 \leq \mu(a) \leq 1$ and $0 \leq v(a) \leq 1$ are a pair of thresholds of a.

Intuitively, the above definition shows the cognitive process between objects and attributes from the fuzzy formal context, fuzzy subset $\widetilde{A}$, membership $\widetilde{I}$ (or nonmembership $\widetilde{I}^{-}$), and thresholds $(\mu(a), v(a))$. Note that Definition 4 can also represent the form of Definitions 1 and 2 by $\mu(a) = v(a) = 0$. Essentially, introducing a pair of parameters can appropriately reduce the extent range of the concept, thereby enhancing the cognitive ability for fuzzy concept ontology in the following concept cognitive process. In addition, the positive and negative cognitive operators satisfy the following properties.

Property 1: Let $(\Omega, \Psi, \widetilde{I}, D, J)$ be a fuzzy decision formal context, $\widetilde{\mathcal{L}}$ and $\mathcal{H}$ be two cognitive operators. For any $X_1, X_2 \subseteq \Omega$, and $\widetilde{A} \in \Gamma^{\Psi}$, the following statements hold:

1) $X_1 \subseteq X_2 \Rightarrow \widetilde{\mathcal{L}}(X_2) \subseteq \widetilde{\mathcal{L}}(X_1)$;
2) $X_1 \subseteq X_2 \Rightarrow \widetilde{\mathcal{L}}^{-}(X_2) \subseteq \widetilde{\mathcal{L}}^{-}(X_1)$;
3) $\widetilde{\mathcal{L}}(X_1 \cup X_2) = \widetilde{\mathcal{L}}(X_2) \cap \widetilde{\mathcal{L}}(X_1)$;
4) $\widetilde{\mathcal{L}}^{-}(X_1 \cup X_2) = \widetilde{\mathcal{L}}^{-}(X_2) \cap \widetilde{\mathcal{L}}^{-}(X_1)$;
5) $\mathcal{H}(\widetilde{A}) = \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}(x)\}$; and
6) $\mathcal{H}^{-}(\widetilde{A}) = \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}^{-}(x)\}$.

Proof: See Appendix I for the proof of Property 1.

Furthermore, we can construct its fuzzy three-way concept form based on its positive and negative cognitive operators in Definition 4. Obviously, in a fuzzy formal context, given an object set or two attribute sets, we can obtain a fuzzy three-way concept according to the cognitive operators in Definitions 3 and 4. It means that the object $x \in \Omega$ can be used to construct an F3WG-concept. Then, the following property holds.

Property 2: Let $(\Omega, \Psi, \widetilde{I}, D, J)$ be a fuzzy decision formal context, for any $x \in \Omega$, $(\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x), (\widetilde{\mathcal{L}}(x), \widetilde{\mathcal{L}}^{-}(x)))$ is a fuzzy three-way concept.

Proof: See Appendix I for the proof of Property 2.

IV. Proposed F3WG-CCL Method

Here, we will first show the proposed method F3WG-CCL, which utilizes F3WG-concept in Section III to initial concept space generation and then discuss the concept-cognitive process under the dynamic environment to cognitive things. In this section, we utilize Fig. 2 to show the overall procedure of the proposed method, which includes three main stages: concept learning, concept cognitive, and concept recognition. The detailed flowchart of the F3WG-CCL is shown in Fig. 2.

A. Initial F3WG-Concept Space Learning

Based on the analysis in Section III, one can learn the fuzzy-granular concept via three-way concept analysis (i.e., F3WG-concept) from the whole object set (i.e., Ω) in a fuzzy formal context. However, in what follows, we learn the F3WG-concept from the local subobject set (i.e., $D_i \in \Omega/D$) and then initial F3WG-concept space generation.

Given a fuzzy decision formal context $(\Omega, \Psi, \widetilde{I}, D, J)$, $\Omega/D = \{D_1, D_2, \ldots, D_l\}$ be a decision division. For any $x \in D_k (k = 1, 2, \ldots, l)$, two pair set-valued mappings $\widetilde{\mathcal{L}}_k : 2^{D_k} \to \Gamma^{\Psi}$ and $\mathcal{H}_k : \Gamma^{\Psi} \to 2^{D_k}$, $\widetilde{\mathcal{L}}_k^{-} : 2^{D_k} \to \Gamma^{\Psi}$, and $\mathcal{H}_k^{-} : \Gamma^{\Psi} \to 2^{D_k}$ are, respectively, called the positive and negative cognitive operators with a local subobject set D_k.

Property 3: Let $(\Omega, \Psi, \widetilde{I}, D, J)$ be a fuzzy decision formal context, for any $D_i \in \Omega/D$ and $x \in D_i$, $(\mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^{-}\widetilde{\mathcal{L}}_k^{-}(x), (\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^{-}(x)))$ is a F3WG-concept.

Proof: See Appendix I for the proof of Property 3.

Intuitively, Property 3 shows that F3WG-concept is an object-oriented fuzzy-granular three-way concept from a local fuzzy decision formal context. It should be pointed out that the object-oriented fuzzy three-way concept (or, fuzzy concept) usually performs well for the classification problem without the need to learn attribute-oriented fuzzy concepts, as illustrated in works [7], [11], and [13].

Definition 5: Let $(\Omega, \Psi, \widetilde{I}, D, J)$ be a fuzzy decision formal context, for any $D_k \in \Omega/D$ and $x \in D_k$, the $(\mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^{-}\widetilde{\mathcal{L}}_k^{-}(x), (\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^{-}(x)))$ is a F3WG-concept. Then, the set of all F3WG-concepts in D_i are represented as follows:

$$\widetilde{\mathcal{G}}^{D_k} = \{(\mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^{-}\widetilde{\mathcal{L}}_k^{-}(x), (\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^{-}(x))) | x \in D_k\}$$

where $\widetilde{\mathcal{G}}^{D_k}$ is referred to as the F3WG-concept subspace of subobject set D_k.

As indicated in Definition 5, we only consider the situation that object-oriented F3WG-concept. Meanwhile, the initial F3WG-concept space $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{G}}^{D_1}, \widetilde{\mathcal{G}}^{D_2}, \ldots, \widetilde{\mathcal{G}}^{D_l}\}$ can be constructed by means of a subobject set $D_k \in \Omega$. The initial F3WG-concept space learning process is shown in Algorithm 1, and its time complexity is $O(|\Omega|^2|\Psi|)$.

B. Concept-Cognitive Process

Considering the information on the object set and attribute set will be updated as time goes by in the real world. Hence, we sign an update mechanism for the proposed CCL system in this section.

Suppose, $\Omega_i^D = \{\Omega_i^{D_1}, \Omega_i^{D_2}, \ldots, \Omega_i^{D_l}\}$ is referred to as the object set under the ith cognitive state. For brevity, for any $D_k \subseteq D$, the s object sets $\Omega_1^{D_k}, \Omega_2^{D_k}, \ldots, \Omega_s^{D_k}$ with $\Omega_1^{D_k} \subseteq \Omega_2^{D_k} \subseteq \ldots \subseteq \Omega_s^{D_k}$ are denoted by $\{\Omega_t^{D_k}\} \uparrow$ and similarly, the

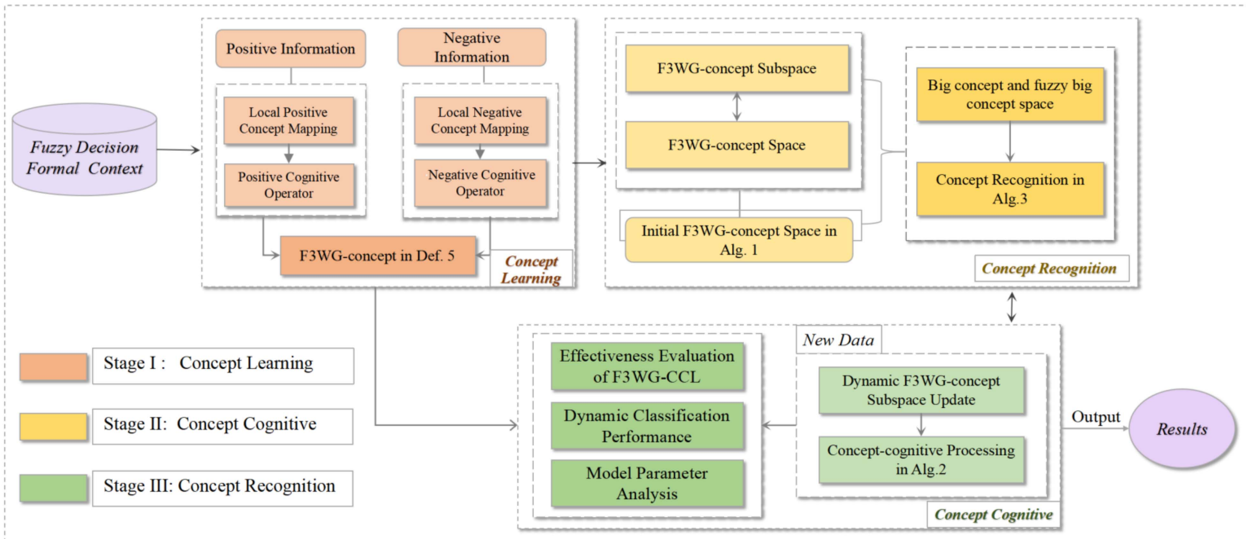


Fig. 2. Processes of the proposed F3WG-CCL method.

Algorithm 1: Initial F3WG-concept Space Learning.

Input: A formal context $(\Omega, \Psi, \widetilde{I}, D, J)$, parameter $\mu(a)$ and $v(a)$;
Output: F3WG-concept space $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{G}}^{D_1}, \widetilde{\mathcal{G}}^{D_2}, \ldots, \widetilde{\mathcal{G}}^{D_l}\}$;
Initial $\widetilde{\mathcal{G}} = \emptyset$;
for $D_k \subseteq \Omega / D$ **do**
  Initial $\widetilde{\mathcal{G}}^{D_k} = \emptyset$;
  for all $x \in D_k$ **do**
    Learn F3WG-concept $(\mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^-\widetilde{\mathcal{L}}_k^-(x), (\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^-(x)))$ according to definition 4 and property 3;
    Construct F3WG-concept subspace $\widetilde{\mathcal{G}}^{D_k} \leftarrow (\mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^-\widetilde{\mathcal{L}}_k^-(x), (\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^-(x)))$ according to definition 5;
  end for
end for
Initial F3WG-concept space $\widetilde{\mathcal{G}} \leftarrow \widetilde{\mathcal{G}}^{D_k}$;
return $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{G}}^{D_1}, \widetilde{\mathcal{G}}^{D_2}, \ldots, \widetilde{\mathcal{G}}^{D_l}\}$.

m attribute sets $\Psi_1, \Psi_2, \ldots, \Psi_m$ with $\Psi_1 \subseteq \Psi_2 \subseteq \ldots \subseteq \Psi_m$ are denoted by $\{\Psi_t\} \uparrow$.

Definition 6: Let $\Omega_{i-1}^{D_k}$ and $\Omega_i^{D_k}$ be object sets of $\{\Omega_t^{D_k}\} \uparrow$, Ψ_{i-1} and Ψ_i be attribute sets of $\{\Psi_t\} \uparrow$; Denote $\Delta\Omega_{i-1}^{D_k} = \Omega_i^{D_k} - \Omega_{i-1}^{D_k}$, $\Delta\Psi_{i-1} = \Psi_i - \Psi_{i-1}$. Suppose the following holds:

1) $\widetilde{\mathcal{L}}_{k,i-1}: 2^{\Omega_{i-1}^{D_k}} \to \Gamma^{\Psi_{i-1}}$, $\mathcal{H}_{k,i-1}: \Gamma^{\Psi_{i-1}} \to 2^{\Omega_{i-1}^{D_k}}$;
2) $\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}: 2^{\Delta\Omega_{i-1}^{D_k}} \to \Gamma^{\Psi_{i-1}}, \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}: \Gamma^{\Psi_{i-1}} \to 2^{\Delta\Omega_{i-1}^{D_k}}$;
3) $\widetilde{\mathcal{L}}_{k,\Delta\Psi_{i-1}}: 2^{\Omega_i^{D_k}} \to \Gamma^{\Delta\Psi_{i-1}}, \mathcal{H}_{k,\Delta\Psi_{i-1}}: \Gamma^{\Delta\Psi_{i-1}} \to 2^{\Omega_i^{D_k}}$;
4) $\widetilde{\mathcal{L}}_{k,i}: 2^{\Omega_i^{D_k}} \to \Gamma^{\Psi_i}$, $\mathcal{H}_{k,i}: \Gamma^{\Psi} \to 2^{\Omega_i^{D_k}}$;
5) $\widetilde{\mathcal{L}}^-_{k,i-1}: 2^{\Omega_{i-1}^{D_k}} \to \Gamma^{\Psi_{i-1}}$, $\mathcal{H}^-_{k,i-1}: \Gamma^{\Psi_{i-1}} \to 2^{\Omega_{i-1}^{D_k}}$;
6) $\widetilde{\mathcal{L}}^-_{k,\Delta\Omega_{i-1}^{D_k}}: 2^{\Delta\Omega_{i-1}^{D_k}} \to \Gamma^{\Psi_{i-1}}, \mathcal{H}^-_{k,\Delta\Omega_{i-1}^{D_k}}: \Gamma^{\Psi_{i-1}} \to 2^{\Delta\Omega_{i-1}^{D_k}}$;
7) $\widetilde{\mathcal{L}}^-_{k,\Delta\Psi_{i-1}}: 2^{\Omega_i^{D_k}} \to \Gamma^{\Delta\Psi_{i-1}}, \mathcal{H}^-_{k,\Delta\Psi_{i-1}}: \Gamma^{\Delta\Psi_{i-1}} \to 2^{\Omega_i^{D_k}}$;
8) $\widetilde{\mathcal{L}}^-_{k,i}: 2^{\Omega_i^{D_k}} \to \Gamma^{\Psi_i}$, $\mathcal{H}^-_{k,i}: \Gamma^{\Psi} \to 2^{\Omega_i^{D_k}}$;

be eight pairs of cognitive operators satisfying the following properties:

$$\widetilde{\mathcal{L}}_{k,i}(x) = \begin{cases} \widetilde{\mathcal{L}}_{k,i-1}(x) \cup \widetilde{\mathcal{L}}_{k,\Delta\Psi_{i-1}}(x), & if\ x \in \Omega_{i-1}^{D_k} \\ \widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x) \cup \widetilde{\mathcal{L}}_{k,\Delta\Psi_{i-1}}(x), & else \end{cases}$$

$$\mathcal{H}_{k,i}(a) = \begin{cases} \mathcal{H}_{k,i-1}(a) \cup \mathcal{H}_{k,\Delta\widetilde{\Omega}_{i-1}^{D_k}}(a), & if\ a \in \Psi_{i-1} \\ \mathcal{H}_{k,\Delta\Psi_{i-1}}(a), & else \end{cases}$$

$$\widetilde{\mathcal{L}}^-_{k,i}(x) = \begin{cases} \widetilde{\mathcal{L}}^-_{k,i-1}(x) \cup \widetilde{\mathcal{L}}^-_{k,\Delta\Psi_{i-1}}(x), & if\ x \in \Omega_{i-1}^{D_k} \\ \widetilde{\mathcal{L}}^-_{k,\Delta\Omega_{i-1}^{D_k}}(x) \cup \widetilde{\mathcal{L}}^-_{k,\Delta\Psi_{i-1}}(x), & else \end{cases}$$

$$\mathcal{H}^-_{k,i}(a) = \begin{cases} \mathcal{H}^-_{k,i-1}(a) \cup \mathcal{H}^-_{k,\Delta\widetilde{\Omega}_{i-1}^{D_k}}(a), & if\ a \in \Psi_{i-1} \\ \mathcal{H}^-_{k,\Delta\Psi_{i-1}}(a), & else \end{cases}$$

where $\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}$, $\widetilde{\mathcal{L}}^-_{k,\Delta\Omega_{i-1}^{D_k}}$, $\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}$, and $\mathcal{H}^-_{k,\Delta\Omega_{i-1}^{D_k}}$ are empty sets, when $\Delta\Omega_{i-1}^{D_k} = \emptyset$; $\widetilde{\mathcal{L}}_{k,\Delta\Psi_{i-1}}$, $\widetilde{\mathcal{L}}^-_{k,\Delta\Psi_{i-1}}$, $\mathcal{H}_{k,\Delta\Psi_{i-1}}$, and $\mathcal{H}^-_{k,\Delta\Psi_{i-1}}$ are empty sets, when $\Delta\Psi_{i-1} = \emptyset$. Then, we say $\widetilde{\mathcal{L}}_{k,i}$ and $\mathcal{H}_{k,i}$ are extended cognitive operators of $\widetilde{\mathcal{L}}_{k,i-1}$ and $\mathcal{H}_{k,i-1}$ with the update information $\Delta\Omega_{i-1}^{D_k}$ and $\Delta\Psi_{i-1}$.

Intrinsically, Definition 6 characterizes updating the knowledge representation of cognitive operators. $\widetilde{\mathcal{L}}_{k,i-1}$ and $\mathcal{H}_{k,i-1}$ can be regarded as the last stage of knowledge expression. Then, $\widetilde{\mathcal{L}}_{k,i}$ and $\mathcal{H}_{k,i}$ can be regarded as the current stage of knowledge expression, which results from updating the last state of knowledge expression with the newly input information $\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}$, $\widetilde{\mathcal{L}}_{k,\Delta\Psi_{i-1}^{D_k}}$, $\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}$, and $\mathcal{H}_{k,\Delta\Psi_{i-1}^{D_k}}$.

Note that the concept-cognitive process was often considered incremental due to the whole being something else than the sum of its part [6], [10], [28]. In what follows, by analyzing the update mechanism of the cognitive operator. The F3WG-concept cognitive mechanism with object updated is investigated as follows.

Property 4: Let $\Omega_i^{D_k}$ be a object set about D_k under ith cognitive state. For any object $x \in \Omega_i^{D_k}$, then the following statements hold:

1) for any $x \in \Omega_i^{D_k}$, if $x \in \Omega_{i-1}^{D_k}$, then

$$\begin{aligned}&(\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}^{-}_{k,i}\widetilde{\mathcal{L}}^{-}_{k,i}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}^{-}_{k,i}(x)))\\ &= ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}^{-}_{k,i-1}\widetilde{\mathcal{L}}^{-}_{k,i-1}(x))\\ &\cup (\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}^{-}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}^{-}_{k,i-1}(x)),\\ &(\widetilde{\mathcal{L}}_{k,i-1}(x), \widetilde{\mathcal{L}}^{-}_{k,i-1}(x))).\end{aligned}$$

2) for any $x \in \Omega_i^{D_k}$, if $x \notin \Omega_{i-1}^{D_k}$, then

$$\begin{aligned}&(\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}^{-}_{k,i}\widetilde{\mathcal{L}}^{-}_{k,i}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}^{-}_{k,i}(x)))\\ &= ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x) \cap \mathcal{H}^{-}_{k,i-1}\widetilde{\mathcal{L}}^{-}_{k,\Delta\Omega_{i-1}^{D_k}}(x))\\ &\cup (\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x) \cap \mathcal{H}^{-}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}^{-}_{k,\Delta\Omega_{i-1}^{D_k}}(x)),\\ &(\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x), \widetilde{\mathcal{L}}^{-}_{k,\Delta\Omega_{i-1}^{D_k}}(x))).\end{aligned}$$

Proof: See Appendix I for the proof of Property 4.

Definition 7: Let $\Omega_i^{D_k}$ be a object set about D_k under ith cognitive state. For any object $x \in \Omega_i^{D_k}$, the object-oriented F3WG-concept space about D_k under ith cognitive state can be defined as follows:

$$\begin{aligned}\widetilde{\mathcal{G}}_i^{D_k} = \{(&\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}^{-}_{k,i}\widetilde{\mathcal{L}}^{-}_{k,i}(x),\\ &\times (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}^{-}_{k,i}(x)))|x \in \Omega_i^{D_k}\}.\end{aligned}$$

Based on the above discussion, one can update the F3WG-concept space under ith cognitive states (i.e., $\widetilde{\mathcal{G}}_i = \{\widetilde{\mathcal{G}}_i^{D_1}, \widetilde{\mathcal{G}}_i^{D_2}, \ldots, \widetilde{\mathcal{G}}_i^{D_l}\}$) through our concept-cognitive process. The details are shown in Algorithm 2, and its time complexity is $O(|\Omega_i||\Omega_{i-1}||\Psi|)$.

So far, a concept-cognitive process of the proposed method F3WG-CCL is completed, and it is obvious that all F3WG-CCL have been learned, and the concept space has been updated at different cognition stages. However, in the classification problem, although using concept space to replace the concept lattice can improve the efficiency of concept learning, it still has concept redundancy, especially in the fuzzy formal context. Next, we will introduce concept recognition based on big concept priority to complete the proposed method.

C. Concept Recognition Based on Big Concept Priority

As we all know, fuzzy concepts bring different information values and significance for concept recognition. As mentioned above, some studies [7], [9], [11], [13], [14] unify the view of using pseudoconcept for fuzzy concept recognition. At this point, two questions are worth considering: 1) why the pseudoconcept is used and 2) whether the pseudoconcept is interpretable.

Algorithm 2: Concept-cognitive Process.

Input: F3WG-concept subspace $\widetilde{\mathcal{G}}_{i-1}^{D_k}$, the added object set $\Delta\Omega_{i-1}$;
Output: F3WG-concept space $\widetilde{\mathcal{G}}_i^{D_k}$;
Initial $\widetilde{\mathcal{G}}_i = \{\widetilde{\mathcal{G}}_i^{D_1}, \widetilde{\mathcal{G}}_i^{D_2}, \ldots, \widetilde{\mathcal{G}}_i^{D_l}\}$;
for all $x \in \Omega_i^{D_k}$ **do**
  if $x \in \Omega_{i-1}^{D_k}$ **then**
    Update F3WG-concept $(\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}^{-}_{k,i}\widetilde{\mathcal{L}}^{-}_{k,i}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}^{-}_{k,i}(x)))$ according to item 1) of property 4;
  else Update F3WG-concept $(\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}^{-}_{k,i}\widetilde{\mathcal{L}}^{-}_{k,i}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}^{-}_{k,i}(x)))$ according to item 2) of property 4;
  end if
  $\widetilde{\mathcal{G}}_i^{D_k} \leftarrow (\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}^{-}_{k,i}\widetilde{\mathcal{L}}^{-}_{k,i}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}^{-}_{k,i}(x)))$;
end for
return $\widetilde{\mathcal{G}}_i^{D_k}$.

Subsequently, we pondered the following answer: Big concepts are more effective (i.e., big concept priority). Specifically, a pseudoconcept is a big concept in its own right, but its extent and intent cannot refer to each other and lack interpretability in some sense. The fuzzy concept space needs to be simplified by fusing fuzzy concepts with extent inclusion relations.

Hence, we put forward a new view for concept recognition in this section, which can identify concepts effectively and avoid learning pseudoconcepts. Of course, the big concept priority principle mainly points out why previous studies prefer pseudoconcepts. It is not to deny pseudoconcepts but to provide a new viewpoint for other fuzzy-based CCL methods. In order to illustrate and verify this view, we will use this simple way to verify the big concept in the concept recognition effect for the proposed method.

Definition 8: Let $\Omega_i^{D_k}$ be a object set about D_k under ith cognitive state, for all F3WG-concept $(X_{i,j}, (\widetilde{A}_{i,j,1}, \widetilde{A}_{i,j,2})) \in \widetilde{\mathcal{G}}_i^{D_k}$, we say that $(X_{i,b_u}, (\widetilde{A}_{i,b_u,1}, \widetilde{A}_{i,b_u,2}))$ is a big concept if there exists $X_{i,1} \subseteq X_{i,2} \subseteq, \ldots, \subseteq X_{i,b_u} \subseteq \Omega_i^{D_k}$. Then, the fuzzy big concept subspace $\widetilde{\mathcal{BG}}_i^{D_k}$ can be defined as follows:

$$\widetilde{\mathcal{BG}}_i^{D_k} = \{(X_{i,b_u}, (\widetilde{A}_{i,b_u,1}, \widetilde{A}_{i,b_u,2}))|i = 1, 2, \ldots\}$$

where, $i = 1, 2, \ldots, j = 1, 2, \ldots,$ and $b_u = 1, 2, \ldots, j$.

Property 5: Let $\Omega_i^{D_k}$ be a object set about D_k under ith cognitive state, the $\widetilde{\mathcal{BG}}_i^{D_k}$ is a fuzzy big concept subspace about the F3WG-concept subspace $\widetilde{\mathcal{G}}_i^{D_k}$. Then, the following inequality holds:

$$1 \leq |\widetilde{\mathcal{BG}}_i^{D_k}| \leq |\widetilde{\mathcal{G}}_i^{D_k}|.$$

Proof: See Appendix I for the proof of Property 5.

For brevity, $\widetilde{\mathcal{BG}}_i^{D_k}$ is referred to as big concept space of F3WG-concept about $\widetilde{\mathcal{G}}_i^{D_k}$, then we have $\widetilde{\mathcal{BG}}_i =$

Algorithm 3: Concept Recognition via Big Concept Priority.

Input: F3WG-concept space $\widetilde{\mathcal{G}}_i = \{\widetilde{\mathcal{G}}_i^{D_1}, \widetilde{\mathcal{G}}_i^{D_2}, \ldots, \widetilde{\mathcal{G}}_i^{D_l}\}$, the added object set $\Delta\Omega_{i-1}$;
Output: $\mathcal{R}^*_{h,D_k}$ and D^*_k;
Get the big concept space $\widetilde{\mathcal{BG}}_i = \{\widetilde{\mathcal{BG}}_i^{D_1}, \widetilde{\mathcal{BG}}_i^{D_2}, \ldots, \widetilde{\mathcal{BG}}_i^{D_l}\}$ according to definition 8;
for all $x \in \Delta\Omega_{i-1}$ **do**
  Get the data $(x, (\widetilde{A}(x), \widetilde{A}^-(x)))$;
end for
for all $\widetilde{\mathcal{G}}_i^{D_k} \subseteq \widetilde{\mathcal{G}}_i$ **do**
  for all $(X_{h,k}, (\widetilde{A}_{h,k,1}, \widetilde{a}^-_{h,k,2})) \in \widetilde{\mathcal{G}}_i^{D_k}$ **do**
    Compute the recognitive degree $\mathcal{CRE}_{h,D_k}$ according to definition 9;
    $\mathcal{R}^*_{h,D_k} \leftarrow \min \mathcal{CRE}_{h,D_k}$;
    $D^*_k \leftarrow \mathrm{argmin}_{h,D_k} \mathcal{R}^*_{h,D_k}$;
  end for
end for
return $\mathcal{R}^*_{h,D_k}$ and D^*_k.

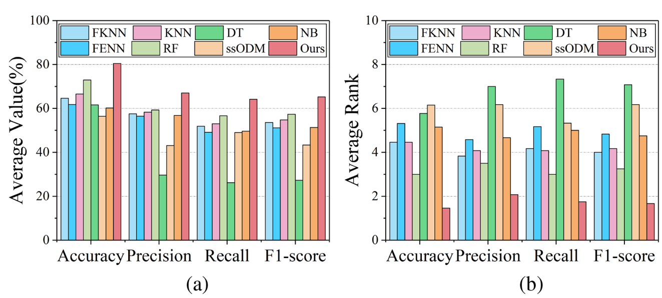


Fig. 3. Average classification performance of F3WG-CCL and other fuzzy classification methods. (a) Shows average value and (b) shows average rank of compared methods on 12 datasets.

$\{\widetilde{\mathcal{BG}}_i^{D_1}, \widetilde{\mathcal{BG}}_i^{D_2}, \ldots, \widetilde{\mathcal{BG}}_i^{D_l}\}$. Furthermore, we can define a concept recognition based on big concept priority.

Definition 9: Let $\Delta\Omega_{i-1} = \Omega_i - \Omega_{i-1}$ be a new object set under ith cognitive state, for $x \in \Delta\Omega_{i-1}$, $(x, (\widetilde{A}(x), \widetilde{A}^-(x)))$ be a new data. For any $(X, (\widetilde{A}_1, \widetilde{A}_2)) \in \widetilde{\mathcal{G}}_i^{D_k}$, the concept recognition degree between the two can be defined by

$$\mathcal{CRE} = \sqrt{\|\widetilde{A}_1 - \widetilde{A}(x)\|^2 + \|\widetilde{A}_2 - \widetilde{A}^-(x)\|^2}.$$

The smaller the value of $\mathcal{CRE}$, the stronger the relationship between the two. Therefore, for any added object x, from Definition 9, we can compute the recognition degree between x and any concept in $\widetilde{\mathcal{G}}_i^{D_k}$. In addition, we can compute the global minimum concept recognition degree between the new data $(x, (\widetilde{A}(x), \widetilde{A}^-(x)))$ and $\widetilde{\mathcal{G}}_i$, denoted by $\mathcal{R}^* = \min \mathcal{CRE}$. The details are shown in Algorithm 3, and its time complexity is $O(|\Omega|(|\Omega|^2 + |\Psi|))$.

TABLE I
BASIC DESCRIPTION OF SELECTED DATASETS

No.s	Dataset	Object	Feature	Class
D1	BreastTissue	106	10	6
D2	Parkinsons	195	23	2
D3	Seeds	210	8	3
D4	Hill	606	101	2
D5	BreastCancer	683	10	2
D6	Mice Protein Expression	1077	68	8
D7	Cardiotocography	2126	23	3
D8	Abalone	4177	9	3
D9	Spambase	4597	57	2
D10	Ring	7400	20	2
D11	EGS	10000	14	2
D12	Nursery	12960	9	5

V. EXPERIMENTS

In the subsequent experiments, we aim to answer the following research questions (Rqs).

1) Rq.1. Does the proposed F3WG-CCL model outperform related state-of-the-art peers?
2) Rq.2. What are the differences between F3WG-CCL and other CCL mechanisms?
3) Rq.3. How do our new standpoints influence the performance of F3WG-CCL?

A. General Settings

In this section, twelve public datasets from UCI Machine Learning Repository[1] are selected to demonstrate the performance of the proposed method compared with others. The detailed information about datasets is shown in Table I, and they are first fuzzified according to the following equation:

$$\widetilde{I}(x, a) = \frac{f(x, a) - \min(f(a))}{\max(f(a)) - \min(f(a))}$$

where the $f(x, a)$ denotes the value of object x under feature a, the $\max(f(a))$ and $\min(f(a))$ are the maximum and minimum values of all objects on feature a.

In order to illustrate the effectiveness of the proposed method, We compare F3WG-CCL with nine related state-of-the-art fuzzy classification methods, including dynamic updating mechanism of progressive weighted fuzzy concept (DM-PWFC) [14], incremental learning mechanism based on progressive fuzzy three-way concept (ILMPFTC) [13], fuzzy-granular concept-cognitive learning (FGA-CCL) (ours), fuzzy K-nearest neighbor (FKNN) [29], fuzzy edited K-nearest neighbor (FENN) [30], K-nearest neighbor (KNN) (K=3) [31], Random Forest (RF) [31], Naive Bayes (NB) [31], Decision Tree (DT) [31], and ssODM [32]. It is noted that 70% data of each dataset is trained as the training set for constructing concept space, and the remaining 30% data is regarded as testing data for evaluating the classification performance of compared methods. Specifically, the testing data is divided into ten equal parts and added to the testing set to verify the dynamic classification

[1][Online]. Available: http://archive.ics.uci.edu/ml/datasets.php

TABLE II
CLASSIFICATION ACCURACY (%) COMPARISON WITH OTHER SEVEN CLASSIFICATION METHODS ON 12 DATASETS

No.s	FKNN	FENN	KNN	RF	DT	ssODM	NB	Our
D1	58.77$_{\pm 1.02}$	60.93$_{\pm 1.04}$	70.08$_{\pm 1.88}$	63.01$_{\pm 2.23}$	58.69$_{\pm 2.51}$	54.01$_{\pm 2.89}$	68.76$_{\pm 1.97}$	**75.40**$_{\pm 1.78}$
D2	47.68$_{\pm 1.94}$	45.77$_{\pm 1.90}$	47.68$_{\pm 1.94}$	53.99$_{\pm 1.82}$	50.17$_{\pm 1.27}$	44.28$_{\pm 2.92}$	55.59$_{\pm 1.27}$	**83.19**$_{\pm 1.18}$
D3	85.16$_{\pm 0.61}$	86.72$_{\pm 0.58}$	85.32$_{\pm 0.60}$	85.80$_{\pm 0.98}$	81.65$_{\pm 0.96}$	66.17$_{\pm 2.09}$	81.40$_{\pm 0.74}$	**95.90**$_{\pm 0.42}$
D4	62.41$_{\pm 0.26}$	44.68$_{\pm 0.69}$	60.83$_{\pm 0.36}$	72.27$_{\pm 0.82}$	47.91$_{\pm 0.25}$	41.38$_{\pm 2.28}$	32.25$_{\pm 0.82}$	**81.31**$_{\pm 1.78}$
D5	99.67$_{\pm 0.05}$	99.42$_{\pm 0.05}$	99.21$_{\pm 0.08}$	98.91$_{\pm 0.18}$	95.79$_{\pm 0.26}$	99.64$_{\pm 0.05}$	99.12$_{\pm 0.07}$	**99.83**$_{\pm 0.03}$
D6	38.22$_{\pm 1.42}$	38.22$_{\pm 1.42}$	39.15$_{\pm 1.50}$	51.63$_{\pm 1.49}$	20.28$_{\pm 0.57}$	22.33$_{\pm 1.57}$	40.77$_{\pm 1.18}$	**59.32**$_{\pm 1.30}$
D7	61.58$_{\pm 0.29}$	59.65$_{\pm 0.37}$	61.10$_{\pm 0.34}$	66.49$_{\pm 0.40}$	61.10$_{\pm 0.75}$	28.83$_{\pm 1.62}$	37.27$_{\pm 1.05}$	**73.92**$_{\pm 1.36}$
D8	53.70$_{\pm 0.34}$	52.17$_{\pm 0.49}$	53.17$_{\pm 0.26}$	60.00$_{\pm 0.67}$	28.90$_{\pm 0.28}$	57.47$_{\pm 1.66}$	29.13$_{\pm 1.71}$	**60.56**$_{\pm 0.37}$
D9	82.36$_{\pm 0.49}$	81.88$_{\pm 0.49}$	81.55$_{\pm 0.48}$	91.02$_{\pm 0.45}$	78.01$_{\pm 0.51}$	87.00$_{\pm 0.47}$	32.73$_{\pm 1.05}$	**92.17**$_{\pm 0.44}$
D10	37.33$_{\pm 1.62}$	23.75$_{\pm 1.95}$	50.89$_{\pm 1.21}$	93.05$_{\pm 0.35}$	81.31$_{\pm 0.22}$	73.00$_{\pm 0.50}$	**96.62**$_{\pm 0.11}$	88.99$_{\pm 0.12}$
D11	83.78$_{\pm 0.37}$	82.81$_{\pm 0.45}$	85.02$_{\pm 0.33}$	99.86$_{\pm 0.01}$	**100.0**$_{\pm 0.00}$	95.57$_{\pm 0.11}$	98.58$_{\pm 0.02}$	87.29$_{\pm 0.38}$
D12	64.67$_{\pm 0.90}$	65.07$_{\pm 0.97}$	64.16$_{\pm 0.93}$	38.85$_{\pm 2.21}$	35.71$_{\pm 2.41}$	7.13$_{\pm 1.17}$	49.76$_{\pm 2.00}$	**67.68**$_{\pm 1.53}$
Ave.	64.61$_{\pm 0.78}$	61.76$_{\pm 0.87}$	66.51$_{\pm 0.82}$	72.91$_{\pm 0.97}$	61.63$_{\pm 0.83}$	56.4$_{\pm 1.44}$	60.17$_{\pm 1.00}$	**80.46**$_{\pm 0.89}$
Rank	4.46	5.31	4.46	3.00	5.77	6.15	5.15	**1.46**

The best results are highlighted in bold entities.

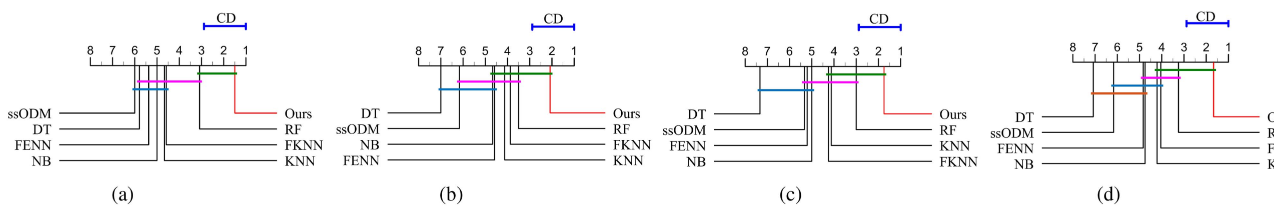


Fig. 4. Nemenyi test on F3WG-CCL and seven other methods.Results between these methods in (a) accuracy, (b) precision, (c) recall, and (d) F1-score.

performance. Meanwhile, to reduce the randomness of the experiment, we still ran ten times on each time node to obtain the average results.

B. Performance Comparison (Rq.1)

To further demonstrate the superiority of F3WG-CCL, in this part, we compare it with seven other popular fuzzy classification methods from classification accuracy, precision, recall, and F1-score, as shown in Fig. 3. Table II only records the detailed classification accuracy on 12 datasets, with the optimal results in bold.

From the classification accuracy, the F3WG-CCL performs better than the other seven popular classification methods in ten datasets, except Datasets 10 and 12. Meanwhile, the F3WG-CCL obtains the highest average accuracy (80.46) and minimum average rank (1.46), significantly better than other compared methods in classification performance. Moreover, the F3WG-CCL achieves the optimal classification results of precision, recall, and F1-score for 9, 10, and 10 times in 12 datasets. The average classification performance of these compared methods is also shown in Fig. 3. The above observations show that F3WG-CCL has apparent advantages compared with the selected methods.

In order to evaluate whether there exists a statistical difference in classification performance between different methods, Friedman's test [33] and Nemenyi's post hoc test [34] are adopted to make the test at a significance level of $P = 0.1$. The null hypothesis for the statistical tests is that the classification performance of compared methods is the same, and it could be rejected when the tested P-value is smaller than the significance level. When comparing F3WG-CCL with seven selected popular fuzzy classification methods, the test P-values

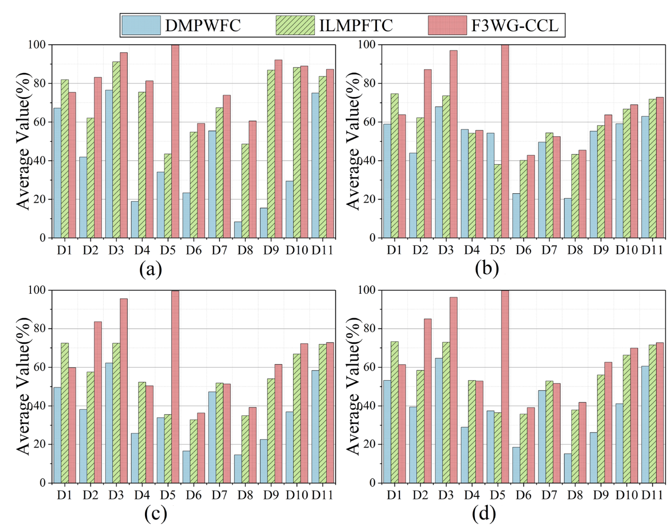


Fig. 5. Average classification performance of three CCLs on 12 datasets. (a) Accuracy, (b) precision, (c) recall, and (d) F1-score.

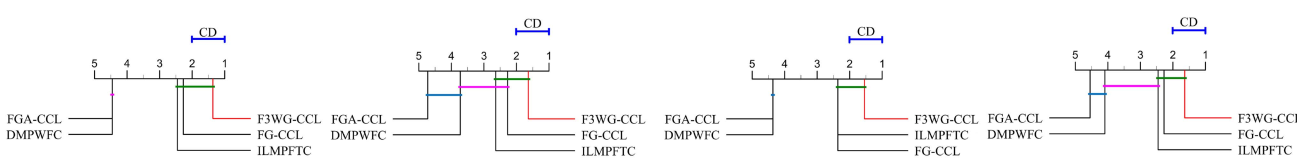
of Friedman test in accuracy, precision, recall, and F1-score are $3.98 \times 10^{-5}, 2.76 \times 10^{-5}, 1.91 \times 10^{-6}$, and 1.90×10^{-6} all smaller than 0.1, which shows a statistically significant difference among these methods. Hence, Nemenyi's post hoc test is adopted and the test results are shown in Fig. 4. In particular, the critical difference (CD) value is 2.78 on 12 datasets when comparing eight methods at $P = 0.1$, which can be computed as follows:

$$\mathrm{CD} = q_P \sqrt{\frac{k(k+1)}{6N}}$$

TABLE III
CLASSIFICATION ACCURACY OF THREE CONCEPT-COGNITIVE METHODS IN DYNAMIC ENVIRONMENT ON 12 DATASETS

No.s	Method	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	$Ave_{\pm Std}$
D1	DMPWFC	50.00	75.00	83.33	75.00	70.00	75.00	71.43	62.50	55.56	55.00	$67.28_{\pm 10.91}$
	ILMPFTC	**100.00**	**100.00**	**100.00**	**87.50**	**80.00**	**83.33**	**78.57**	**68.75**	**61.11**	**60.00**	**$81.93_{\pm 15.33}$**
	F3WG-CCL	**100.00**	**100.00**	**100.00**	75.00	70.00	66.67	64.29	62.50	55.56	60.00	$75.40_{\pm 17.77}$
D2	DMPWFC	40.00	30.00	26.67	20.00	32.00	43.33	48.57	55.00	60.00	64.00	$41.96_{\pm 14.87}$
	ILMPFTC	40.00	40.00	60.00	50.00	60.00	66.67	**71.43**	**75.00**	**77.78**	**80.00**	$62.09_{\pm 14.80}$
	F3WG-CCL	**100.00**	**100.00**	**100.00**	**75.00**	**76.00**	**76.67**	**71.43**	**75.00**	**77.78**	**80.00**	**$83.19_{\pm 11.80}$**
D3	DMPWFC	**100.00**	75.00	83.33	79.17	80.00	72.22	66.67	68.75	68.52	71.67	$76.53_{\pm 9.90}$
	ILMPFTC	**100.00**	91.67	94.44	91.67	90.00	91.67	92.86	87.50	85.19	86.67	$91.17_{\pm 4.26}$
	F3WG-CCL	**100.00**	**100.00**	**100.00**	**100.00**	**96.67**	**94.44**	**95.24**	**93.75**	**88.89**	**90.00**	**$95.90_{\pm 4.20}$**
D4	DMPWFC	5.56	2.78	1.85	1.39	1.11	16.67	28.57	37.50	44.44	49.44	$18.93_{\pm 19.38}$
	ILMPFTC	**100.00**	86.11	87.04	87.50	88.89	76.85	66.67	59.03	53.70	48.89	$75.47_{\pm 17.33}$
	F3WG-CCL	94.44	**97.22**	**98.15**	**95.83**	**96.67**	**82.41**	**73.02**	**64.58**	**58.02**	**52.78**	**$81.31_{\pm 17.83}$**
D5	DMPWFC	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**98.75**	**98.89**	**99.00**	**$99.66_{\pm 0.54}$**
	ILMPFTC	72.41	48.28	44.83	39.66	33.10	39.08	39.41	39.22	40.23	38.62	$43.48_{\pm 10.92}$
	F3WG-CCL	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	98.12	97.22	95.50	$99.08_{\pm 1.60}$
D6	DMPWFC	45.16	24.19	16.13	12.10	10.32	15.05	23.04	23.79	31.90	31.61	$23.33_{\pm 10.73}$
	ILMPFTC	74.19	58.06	59.14	52.42	49.68	51.08	49.77	51.61	**49.10**	53.55	$54.86_{\pm 7.61}$
	F3WG-CCL	**90.32**	**72.58**	**63.44**	**55.65**	**52.26**	**52.69**	**51.61**	**52.02**	48.75	**53.87**	**$59.32_{\pm 12.98}$**
D7	DMPWFC	60.32	67.406	53.44	51.19	49.52	50.26	50.57	54.56	57.85	58.89	$55.41_{\pm 5.72}$
	ILMPFTC	61.90	69.84	64.02	64.68	67.62	70.37	67.12	**68.65**	**69.49**	**70.32**	$67.40_{\pm 2.95}$
	F3WG-CCL	**71.43**	**85.71**	**84.66**	**86.90**	**86.67**	**82.54**	**71.88**	62.90	55.91	50.63	**$73.92_{\pm 13.57}$**
D8	DMPWFC	0.00	0.00	0.00	0.00	1.92	2.13	4.34	16.30	25.60	33.04	$8.33_{\pm 12.23}$
	ILMPFTC	49.60	47.60	45.33	45.20	48.00	49.60	52.34	50.50	49.24	48.96	$48.64_{\pm 2.20}$
	F3WG-CCL	**62.40**	**58.40**	**60.00**	**61.20**	**64.32**	**64.93**	**64.80**	**59.40**	**56.00**	**54.16**	**$60.56_{\pm 3.70}$**
D9	DMPWFC	9.49	6.57	4.93	3.94	3.28	15.64	26.19	34.39	40.88	16.15	$16.15_{\pm 13.42}$
	ILMPFTC	94.16	94.89	94.89	91.24	91.39	91.48	84.88	78.56	74.86	72.41	$86.88_{\pm 8.62}$
	F3WG-CCL	**94.89**	**97.08**	**95.86**	**95.26**	**93.72**	**94.16**	**91.55**	**89.05**	**86.29**	**83.80**	**$92.17_{\pm 4.42}$**
D10	DMPWFC	13.57	12.67	14.63	14.59	14.93	29.03	39.11	46.61	52.39	57.10	$29.46_{\pm 17.84}$
	ILMPFTC	**93.21**	**94.34**	**94.12**	**94.80**	**94.66**	**91.18**	85.33	81.50	78.03	75.02	$88.22_{\pm 7.62}$
	F3WG-CCL	87.33	87.33	87.78	88.69	88.87	89.44	**89.79**	**90.33**	**90.15**	**90.23**	**$88.99_{\pm 1.18}$**
D11	DMPWFC	93.65	90.97	91.86	86.62	76.52	69.79	65.36	61.37	58.31	56.29	$75.07_{\pm 14.75}$
	ILMPFTC	84.62	81.44	80.94	81.10	83.34	84.28	84.66	**84.99**	**85.32**	**85.65**	$83.63_{\pm 1.82}$
	F3WG-CCL	**90.97**	**91.81**	**92.20**	**90.22**	**87.89**	**86.12**	**85.00**	83.61	82.83	82.27	**$87.29_{\pm 3.83}$**
D12	DMPWFC	-	-	-	-	-	-	-	-	-	-	-
	ILMPFTC	21.91	40.85	30.07	45.10	56.08	63.40	66.94	62.50	62.40	66.16	$51.54_{\pm 16.14}$
	F3WG-CCL	**65.21**	**36.08**	**48.97**	**61.73**	**69.38**	**74.48**	**76.44**	**79.38**	**81.67**	**83.51**	**$67.69_{\pm 15.27}$**

The best results are highlighted in bold entities.



(a) (b) (c) (d)

Fig. 6. Nemenyi Test on F3WG-CCL and four CCLs. Results between these methods in (a) Accuracy, (b) precision, (c) recall, and (d) F1-score.

where $q_{P=0.1} = 2.78$, $k = 8$, and $N = 12$. The classification performance of any two compared methods significantly differs when the test value exceeds the CD value. Fig. 4 shows that the F3WG-CCL performs best in all classification performances and is significantly better than the other six methods in classification accuracy. According to the above analysis, the proposed F3WG-CCL outperforms the related state-of-the-art methods in classification performance.

C. Differences Between the Fuzzy-Based CCLs (Rq.2)

In the last section, we only verify the superiority of F3WG-CCL over some popular classification methods in classification performance. And then, we compare the F3WG-CCL with two state-of-the-art fuzzy-based CCL methods (dynamic concept learning mechanisms), i.e., DMPWFC and ILMPFTC, to show differences between the fuzzy-based CCLs.

The dynamic classification accuracy of DMPWFC, ILMPFTC, and F3WG-CCL on 12 datasets is shown in Table III, where the last column shows the average accuracy (Ave) and standard deviation (Std). All the optimal classification results are in bold. From this table, we could find that the accuracy of the proposed F3WG-CCL is higher than that of DMPWFC and ILMPFTC in all datasets, except in Datasets 1 and 5. Note that the DMPWFC is invalid in Dataset 12 due to ignoring the concept that does not need updating. In addition, the other average classification performance of different algorithms on 12 datasets is shown in Table IV and Fig. 5. Specifically, F3WG-CCL performs better than other concept methods in average accuracy for 11 datasets except in Dataset 1. F3WG-CCL achieves the highest value nine times in 12 datasets for the average precision index. Moreover, F3WG-CCL performs well at ten datasets of all datasets in average precision except for Dataset 1. Compared with

TABLE IV
AVERAGE CLASSIFICATION PERFORMANCE OF THREE CONCEPT-COGNITIVE METHODS IN DYNAMIC ENVIRONMENT ON 12 DATASETS

No.s	Mechanism	Accuracy	Precision	Recall	F1-score	No.s	Mechanism	Accuracy	Precision	Recall	F1-score
D1	DMPWFC	67.28±1.09	58.93±1.28	49.56±1.54	53.21±1.38	D7	DMPWFC	55.41±0.57	49.63±0.52	47.28±1.15	47.98±0.75
	ILMPFTC	**81.93**±1.53	**74.68**±2.15	**72.53**±2.19	**73.29**±2.15		ILMPFTC	67.40±0.30	**54.40**±0.67	**51.85**±1.14	**52.89**±0.88
	F3WG-CCL	75.40±1.78	63.78±2.70	59.83±2.94	61.43±2.82		F3WG-CCL	**73.92**±1.36	52.47±0.93	51.39±0.18	51.65±0.54
D2	DMPWFC	41.96±1.49	44.01±0.98	38.19±1.56	39.47±1.25	D8	DMPWFC	8.33±1.22	20.49±1.81	14.52±1.79	15.12±1.79
	ILMPFTC	62.09±1.48	62.25±1.11	57.53±2.17	58.43±1.78		ILMPFTC	48.64±0.22	43.27±0.74	34.99±1.52	37.93±1.25
	F3WG-CCL	**83.19**±1.18	**87.18**±0.93	**83.61**±1.36	**85.04**±1.09		F3WG-CCL	**60.56**±0.37	**45.49**±0.91	**39.28**±1.38	**41.82**±1.21
D3	DMPWFC	76.53±0.99	67.94±1.51	62.20±1.92	64.70±1.71	D9	DMPWFC	15.48±1.36	55.33±0.75	22.52±2.50	26.23±2.64
	ILMPFTC	91.17±0.43	73.58±1.68	72.49±1.94	72.95±1.79		ILMPFTC	86.88±0.86	58.23±1.08	54.10±0.99	56.08±1.03
	F3WG-CCL	**95.90**±0.42	**97.03**±0.31	**95.51**±0.47	**96.25**±0.39		F3WG-CCL	**92.17**±0.44	**63.73**±1.78	**61.52**±1.8	**62.59**±1.79
D4	DMPWFC	18.93±1.94	**56.22**±0.88	25.85±2.59	29.02±2.81	D10	DMPWFC	29.46±1.78	59.14±1.04	36.96±2.59	41.21±2.52
	ILMPFTC	75.47±1.73	54.26±1.62	52.32±1.71	**53.20**±1.66		ILMPFTC	88.22±0.76	66.74±1.69	66.86±1.75	66.31±1.60
	F3WG-CCL	**81.31**±1.78	55.69±0.66	**50.44**±0.25	52.88±0.43		F3WG-CCL	**88.99**±0.12	**68.94**±1.97	**72.25**±2.44	**69.88**±2.10
D5	DMPWFC	34.20±1.57	54.29±0.60	33.94±2.52	37.46±2.23	D11	DMPWFC	75.07±1.48	62.98±0.94	58.37±0.85	60.57±0.89
	ILMPFTC	43.48±1.09	38.05±0.63	35.55±0.82	36.57±0.68		ILMPFTC	83.63±0.18	71.87±1.62	71.88±2.12	71.60±1.85
	F3WG-CCL	**99.83**±0.03	**99.89**±0.02	**99.63**±0.07	**99.76**±0.04		F3WG-CCL	**87.29**±0.38	**72.85**±1.62	**72.77**±1.86	**72.74**±1.73
D6	DMPWFC	23.33±1.07	23.01±0.82	16.53±1.11	18.49±0.99	D12	DMPWFC	-	-	-	-
	ILMPFTC	54.86±0.76	40.16±0.66	32.85±1.21	35.79±0.99		ILMPFTC	51.54±1.61	38.36±0.81	30.49±1.86	32.34±1.53
	F3WG-CCL	**59.32**±1.30	**42.86**±0.64	**36.40**±0.97	**39.15**±0.81		F3WG-CCL	**67.68**±1.53	**54.39**±1.82	**47.19**±2.70	**49.59**±2.37

The best results are highlighted in bold entities.

TABLE V
CLASSIFICATION PERFORMANCE BETWEEN F3WG-CCL, FG-CCL, AND FGA-CCL ON 12 DATASETS

No.s	Accuracy			Precision			Recall			F1-score		
	FG-CCL	FGA-CCL	F3WG-CCL	FG-CCL	FGA-CCL	F3WG-CCL	FG-CCL	FGA-CCL	F3WG-CCL	FG-CCL	FGA-CCL	FG-CCL
D1	62.87±2.88	73.28±1.92	**75.40**±1.78	42.22±4.10	54.93±3.25	**63.78**±2.70	53.67±3.35	**61.30**±3.04	59.83±2.94	45.79±3.89	57.61±3.14	61.43±2.82
D2	32.46±2.19	59.17±1.30	**83.19**±1.18	18.45±1.29	60.04±0.79	**87.18**±0.93	34.30±1.92	55.23±1.96	**83.61**±1.36	22.36±1.45	56.22±1.55	85.04±1.09
D3	35.62±1.77	82.19±0.74	**95.90**±0.42	54.56±2.13	62.43±2.01	**97.03**±0.31	35.49±2.13	59.49±2.39	**95.51**±0.47	41.38±2.22	60.65±2.20	96.25±0.39
D4	67.91±1.07	**83.56**±1.72	81.31±1.78	50.76±0.17	**66.30**±1.64	55.69±0.66	44.88±0.73	**57.89**±1.54	50.44±0.25	47.42±0.47	61.60±1.53	52.88±0.43
D5	8.15±1.27	**99.83**±0.03	**99.83**±0.03	4.07±0.64	99.89±0.02	**99.89**±0.02	20.00±2.58	**99.63**±0.07	**99.63**±0.07	6.50±0.98	99.76±0.04	99.76±0.04
D6	13.63±0.80	54.96±0.60	**59.32**±1.30	18.41±1.16	**44.94**±0.62	42.86±0.64	9.22±0.63	**36.04**±1.06	**36.40**±0.97	11.05±0.62	39.62±0.84	39.15±0.81
D7	63.68±1.45	60.54±0.96	**73.92**±1.36	41.11±0.96	46.34±0.63	**52.47**±0.93	47.42±0.26	40.23±1.03	**51.39**±0.18	43.68±0.67	42.80±0.82	51.65±0.54
D8	7.20±1.17	49.61±0.14	**60.56**±0.37	21.14±1.74	44.50±0.82	**45.49**±0.91	13.42±1.72	35.94±1.56	**39.28**±1.38	7.29±0.955	39.01±1.30	41.82±1.21
D9	22.94±1.18	84.18±0.31	**92.17**±0.44	55.02±0.71	60.75±1.42	**63.73**±1.78	26.46±2.41	58.73±2.08	**61.52**±1.80	31.30±2.33	59.48±1.74	62.59±1.79
D10	41.10±1.51	**89.08**±0.12	88.99±0.12	60.75±1.22	69.00±1.97	**68.94**±1.97	44.11±2.60	**72.32**±2.45	72.25±2.44	48.10±2.31	69.95±2.11	69.88±2.10
D11	75.49±0.76	**87.29**±0.38	**87.29**±0.38	70.87±1.64	**72.85**±1.62	**72.85**±1.62	66.74±2.30	**72.77**±1.86	**72.77**±1.86	68.14±1.99	72.74±1.73	72.74±1.73
D12	4.19±0.70	67.68±1.53	**67.68**±1.53	1.64±0.27	**54.39**±1.82	**54.39**±1.82	4.66±0.60	**47.19**±2.70	**47.19**±2.70	2.24±0.36	49.59±2.37	49.59±2.37
Ave.	36.27±1.39	74.28±0.81	**80.46**±0.89	36.58±1.34	61.36±1.38	**67.03**±1.19	33.36±1.77	58.06±1.81	**64.15**±1.37	31.27±1.52	59.09±1.61	65.23±1.28

The best results are highlighted in bold entities.

DMPWFC and ILMPFTC, F3WG-CCL also obtains the maximum value of the average F1-score 9 times except for Datasets 1, 4, and 7. All the above results illustrate the superiority of the fuzzy-granular concept compared with DMPWFC and ILMPFTC in a dynamic environment.

In order to evaluate whether there exists a statistical difference in classification performance between several fuzzy-based CCLs (note that fuzzy-granular concept-cognitive learning (FG-CCL) and FGA-CCL are two mechanisms of our proposed and also are discussed in the following section), Friedman's test and Nemenyi's post hoc test are adopted to make the test at a significance level of $P = 0.1$. When comparing F3WG-CCL with other CCLs, the test P-values of Friedman test in accuracy, precision, recall, and F1-score are $2.19 \times 10^{-5}, 2.19 \times 10^{-5}, 7.00 \times 10^{-6}$, and 1.36×10^{-5}, lowering than 0.1 and indicating that there is a statistically significant difference between F3WG-CCL and the four classical CCL methods. Note that the DMPWFC is invalid in Dataset 12. Thus, the statistical test in CCL mechanisms is carried out on other 11 datasets. To further test the difference between F3WG-CCL and each other CCL mechanism, Nemenyi's post hoc test is conducted, and the value of the CD is 1.6585 at the confidence level of $P = 0.1$. From Fig. 6, we know the F3WG-CCL ranks first among these CCL methods in classification performances and is significantly better than FGA-CCL and DMPWFC.

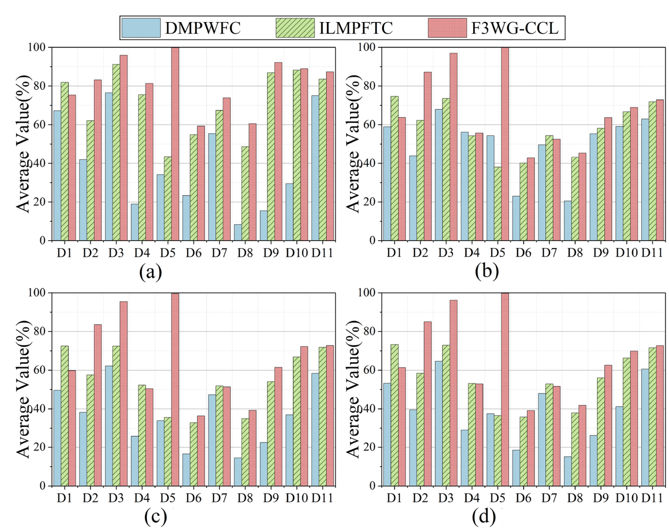


Fig. 7. Average classification performance of three cases on 12 datasets. (a) Accuracy, (b) precision, (c) recall, and (d) F1-score.

D. Influence of the Proposed Standpoints (Rq.3)

To test the influence of our proposed standpoints, we verify the rationality of the proposed view from three aspects: 1) verifying the three-way concept is more effective in a fuzzy

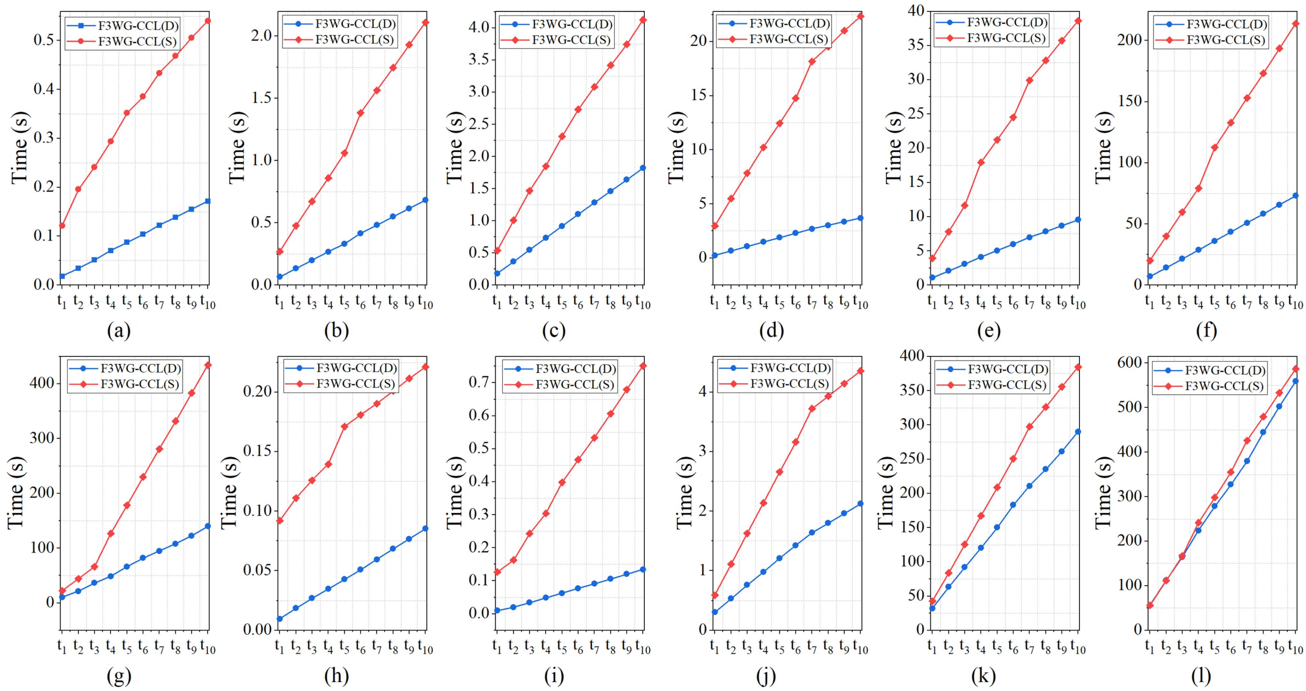


Fig. 8. Time consumption of F3WG-CCL(D) and F3WG-CCL(S). (a) D1, (b) D2, (c), D3, (d) D4, (e) D5, (f) D6, (g) D7, (h) D8, (i) D9, (j) D10, (k) D11, (l) D12.

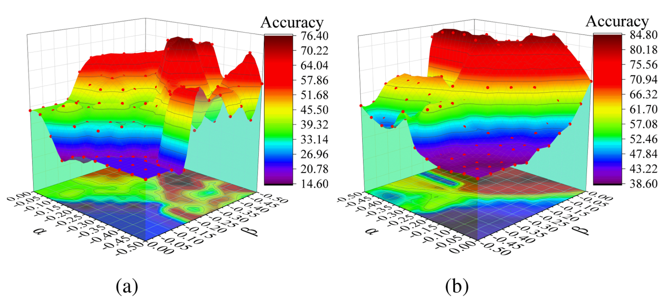


Fig. 9. Average accuracy comparison under different parameters. (a) Dataset 1, (b) Dataset 2.

context for classification tasks, 2) showing that big concepts are more valuable for concept recognition, and 3) the update mechanism is effective.

1) F3WG-concept is Practical: As Section III mentions, the F3WG-concept constructs the fuzzy concept by combing positive and negative information, providing a more comprehensive method to characterize the relationship between objects and features. To demonstrate the superiority of the three-way concept in dynamic classification, we record the average results of F3WG-CCL and FG-CCL (without the three-way concept in F3WG-CCL) in Table V. This table shows that F3WG-CCL performs better than FG-CCL in all experimented datasets, and references [7], [9], [13] also confirm fuzzy three-way concept can show superior performance in classification problems, fully proving the persuasiveness of the three-way concept in the fuzzy context.

2) Big Concept Priority Principle is Reasonable: Moreover, to illustrate the rationality of the principle of big concept priority, we further compare the F3WG-CCL with FGA-CCL (i.e., use all F3WG-concepts for concept recognition), which designs the classification mechanism based on all fuzzy concepts. The dynamic average classification performance of FGA-CCL and F3WG-CCL is shown in Table V. Compared with FGA-CCL, the proposed F3WG-CCL mechanism achieves better classification performance in most datasets, which can be reflected in the maximum average values of F3WG-CCL in accuracy, precision, recall, and F1-score. Fig. 7 shows the effectiveness and rationality of the F3WG concept and big concept priority principle in dynamic classification.

3) Update Mechanism of F3WG-CCL is Effective: To efficiently update the F3WG-CCL-concept space, we give the update mechanism in our methods and display the process in Algorithm 2. To demonstrate the superiority of the update mechanism, we record and analyze the time consumption of these two mechanisms for updating fuzzy concept space at different times in this section. For convenience, the dynamic update mechanism [i.e., F3WG-CCL(D)] and corresponding static mechanism [i.e., F3WG-CCL(S)] are shown in Fig. 8, where the units of consuming time are seconds (s).

Compared with the static mechanism, the dynamic mechanism, i.e., F3WG-CCL, consumes less time, and the average dynamic concept learning time is significantly lower than that of the static mechanism. Moreover, it can also be seen from Fig. 8, the time gap between the two mechanisms becomes more significant with the increase of objects when facing dynamic updates, especially in large-scale data. These results show and answer the question of the effectiveness of the update mechanism in dynamic classification.

4) Parameters Analysis: It is clear that μ and v are two essential parameters used to learn F3WG-concept. Thus, the changes in their values impact the construction of concept space and cause the variation of classification performance of F3WG-CCL as well. Considering the differences between different features, for each feature a, the parameter $\mu(a)$ and $v(a)$ is set to $\mu(a) = \sigma(a) + \alpha$ and $v(a) = \sigma(a^-) + \beta$, where the $\sigma(a)$ and $\sigma(a^-)$ represent the standard deviation of feature a under fuzzy decision formal context. To observe the sensitivity of F3WG-CCL, the α and β are gradually adjusted from –0.5 to 0 with a step of 0.05, and the optimal parameters are determined according to the highest classification accuracy. In addition, the parameters of other compared methods are set consistent with their references.

To comprehensively investigate the effect of these parameters on classification performance, we display the classification accuracy of F3WG-CCL under different parameters on Datasets 1 and 2, in Fig. 9. It can be obtained from these subfigures that the average classification accuracy of F3WG-CCL fluctuates significantly with the change of α and β on all experimental datasets. The experimental results verify that the classification performance of F3WG-CCL has a specific sensitivity of the values of α and β. Meanwhile, the F3WG-CCL achieves optimal classification accuracy in each dataset with different parameters combination. Therefore, the selection process of parameters to obtain the highest classification performance is necessary for the actual application.

VI. Conclusion

This article presents F3WG-CCL for dynamic knowledge discovery in a fuzzy formal context. It adopts threefold ideas: 1) F3WG concept to describe fuzzy ontology, 2) update mechanism to improve the efficiency of concept learning, and 3) big concepts are used for concept recognition. Theoretical and empirical studies show that F3WG-CCL obtains superior concept learning and classification performance.

The current article studies the fuzzy-based cognitive-cognitive learning model by defining F3WG-concept and big concepts for dynamic knowledge discovery. Consequently, some limitations still need to be considered, such as adaptive learning of F3WG-concept and accurate recognition of big concepts. Although our method can significantly enhance the efficiency of concept learning and classification performance for dynamic fuzzy data, it still cannot be learned for billions of data. Hence, how to combine machine learning and deep learning theory into CCL theory also deserves to be explored. We plan to address these challenging problems in future work.

Acknowledgment

The authors would like to thank the Associate Editor and the reviewers for their insightful comments and suggestions.

Appendix

Property 6: Let $(\Omega, \Psi, \widetilde{I}, D, J)$ be a fuzzy decision formal context, $\widetilde{\mathcal{L}}$ and $\mathcal{H}$ be two cognitive operators. For any $X_1, X_2 \subseteq \Omega$ and $\widetilde{A} \in \Gamma^{\Psi}$, the following statements hold.

1) $X_1 \subseteq X_2 \Rightarrow \widetilde{\mathcal{L}}(X_2) \subseteq \widetilde{\mathcal{L}}(X_1)$;
2) $X_1 \subseteq X_2 \Rightarrow \widetilde{\mathcal{L}}^-(X_2) \subseteq \widetilde{\mathcal{L}}^-(X_1)$;
3) $\widetilde{\mathcal{L}}(X_1 \cup X_2)=\widetilde{\mathcal{L}}(X_2) \cap \widetilde{\mathcal{L}}(X_1)$;
4) $\widetilde{\mathcal{L}}^-(X_1 \cup X_2)=\widetilde{\mathcal{L}}^-(X_2) \cap \widetilde{\mathcal{L}}^-(X_1)$;
5) $\mathcal{H}(\widetilde{A}) = \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}(x)\}$;
6) $\mathcal{H}^-(\widetilde{A}) = \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}^-(x)\}$.

Proof:

1) According to definition 2, for any $a \in \Psi$, if $X_1 \subseteq X_2$ we have $\bigwedge_{x\in X_2} \widetilde{I}(x,a) \leq \bigwedge_{x \in X_1} \widetilde{I}(x,a)$, thus $\widetilde{\mathcal{L}}(X_2) \subseteq \widetilde{\mathcal{L}}(X_1)$ holds when $X_1 \subseteq X_2$.
2) Similarly to 1), from definition 3, for any $a \in \Psi$, $\bigwedge_{x\in X_2} \widetilde{I}^-(x,a) \leq \bigwedge_{x \in X_1} \widetilde{I}^-(x,a)$ holds when $X_1 \subseteq X_2$ and then $\widetilde{\mathcal{L}}^-(X_2) \subseteq \widetilde{\mathcal{L}}^-(X_1)$ is obtained.
3) According to definition 2, for any $a \in \Psi$, we have
$$\begin{aligned}\widetilde{\mathcal{L}}&(X_1 \cup X_2)(a)\\ &= \bigwedge_{x \in X_1 \cup X_2} \widetilde{I}(x,a)\\ &= \bigwedge_{x\in X_1} \widetilde{I}(x,a) \bigwedge \bigwedge_{x \in X_2} \widetilde{I}(x,a)\\ &= \widetilde{\mathcal{L}}(X_1)(a) \bigwedge \widetilde{\mathcal{L}}(X_2)(a).\end{aligned}$$
Then we have $\widetilde{\mathcal{L}}(X_1 \cup X_2) = \widetilde{\mathcal{L}}(X_2) \cap \widetilde{\mathcal{L}}(X_1)$.
4) Similarly to 3), according to definition 3, for any $a \in \Psi$, we obtain
$$\begin{aligned}\widetilde{\mathcal{L}}^-&(X_1 \cup X_2)(a)\\ &= \bigwedge_{x \in X_1 \cup X_2} \widetilde{I}^-(x,a)\\ &= \bigwedge_{x\in X_1} \widetilde{I}^-(x,a) \bigwedge \bigwedge_{x \in X_2} \widetilde{I}^-(x,a)\\ &= \widetilde{\mathcal{L}}^-(X_1)(a) \bigwedge \widetilde{\mathcal{L}}^-(X_2)(a).\end{aligned}$$
Thus, $\widetilde{\mathcal{L}}(X_1 \cup X_2) = \widetilde{\mathcal{L}}^-(X_2) \cap \widetilde{\mathcal{L}}^-(X_1)$ holds.
5) According to definition 2, for $\widetilde{A} \in \Gamma^{\Psi}$, we have
$$\begin{aligned}\mathcal{H}(\widetilde{A}) &= \{x \in \Omega | \widetilde{A}(a) \leq \widetilde{I}(x,a), \forall a \in \Psi\}\\ &= \{x \in \Omega | \widetilde{A}(a) \leq \bigwedge_x \widetilde{I}(x,a), \forall a \in \Psi\}\\ &= \left\{x \in \Omega | \widetilde{A}(a) \leq \widetilde{\mathcal{L}}(x)(a), \forall a \in \Psi\right\}\\ &= \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}(x)\}.\end{aligned}$$
Hence, $\mathcal{H}(\widetilde{A}) = \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}(x)\}$ holds.
6) Similarly to 5), according to definition 3, we have
$$\begin{aligned}\mathcal{H}^-(\widetilde{A}) &= \{x \in \Omega | \widetilde{A}(a) \leq \widetilde{I}^-(x,a), \forall a \in \Psi\}\\ &= \left\{x \in \Omega | \widetilde{A}(a) \leq \bigwedge_x \widetilde{I}^-(x,a), \forall a \in \Psi\right\}\\ &= \{x \in \Omega | \widetilde{A}(a) \leq \widetilde{\mathcal{L}}^-(x)(a), \forall a \in \Psi\}\\ &= \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}^-(x)\}.\end{aligned}$$

Therefore, $\mathcal{H}^{-}(\widetilde{A}) = \{x \in \Omega | \widetilde{A} \subseteq \widetilde{\mathcal{L}}^{-}(x)\}$ holds. #

Property 7: Let $(\Omega, \Psi, \widetilde{I}, D, J)$ be a fuzzy decision formal context, for any $x \in \Omega$, $(\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x), (\widetilde{\mathcal{L}}(x), \widetilde{\mathcal{L}}^{-}(x)))$ is a fuzzy three-way concept.

Proof: To prove this Property, we divide it into two steps:
1) prove $\mathcal{H}^{\nabla}(\widetilde{\mathcal{L}}(x), \widetilde{\mathcal{L}}^{-}(x)) = \mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)$ is valid;
2) prove $\widetilde{\mathcal{L}}^{\nabla}(\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)) = (\widetilde{\mathcal{L}}(x), \widetilde{\mathcal{L}}^{-}(x))$ is valid.

- For 1), it is directly obtained from definition 4.
- For 2), according to definition 4, for any $a \in \Psi$, we have

$$\begin{aligned}
&\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)\\
&\quad = \{y \in \Omega | \widetilde{I}(y,a) \geq \widetilde{I}(x,a) + \mu(a)\}\\
&\quad \cap \{y \in \Omega | \widetilde{I}^{-}(y,a) \geq \widetilde{I}^{-}(x,a) + v(a)\}\\
&\quad = \{y \in \Omega | \widetilde{I}(y,a) \geq \widetilde{I}(x,a) + \mu(a)\}\\
&\quad \cap \{y \in \Omega | 1 - \widetilde{I}^{-}(y,a) \leq 1 - \widetilde{I}^{-}(x,a) - v(a)\}\\
&\quad = \{y \in \Omega | \widetilde{I}(y,a) \geq \widetilde{I}(x,a) + \mu(a)\}\\
&\quad \cap \{y \in \Omega | \widetilde{I}(y,a) \leq \widetilde{I}(x,a) - v(a)\}\\
&\quad = \{y \in \Omega | \widetilde{I}(x,a) + \mu(a) \leq \widetilde{I}(y,a) \leq \widetilde{I}(x,a) - v(a)\}.
\end{aligned}$$

Therefore, for any $a \in \Psi$, we have

$$\widetilde{\mathcal{L}}(\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)) = \bigwedge_{x \in \mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)} (\widetilde{I}(x,a) + \mu(a)) = \widetilde{\mathcal{L}}(x).$$

At the same time, we obtain

$$\begin{aligned}
&\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)\\
&\quad = \{y \in \Omega | \widetilde{I}^{-}(x,a) + v(a) \leq \widetilde{I}^{-}(y,a)\\
&\quad \leq \widetilde{I}^{-}(x,a) - \mu(a)\}.
\end{aligned}$$

Then, for any $a \in \Psi$, we have

$$\widetilde{\mathcal{L}}^{-}(\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)) = \bigwedge_{x \in \mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)} (\widetilde{I}^{-}(x,a) + v(a)) = \widetilde{\mathcal{L}}^{-}(x).$$

Hence, we have

$$\widetilde{\mathcal{L}}^{\nabla}(\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x)) = (\widetilde{\mathcal{L}}(x), \widetilde{\mathcal{L}}^{-}(x)).$$

By combining 1) and 2), this property is proven. #

Property 8: Let $(\Omega, \Psi, \widetilde{I}, D, J)$ be a fuzzy decision formal context, for any $D_i \in \Omega/D$ and $x \in D_i$, $(\mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^{-}\widetilde{\mathcal{L}}_k^{-}(x), (\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^{-}(x)))$ is a F3WG-concept.

Proof: To prove this theorem, it is necessary to prove that
1) $\mathcal{H}^{\nabla}(\mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^{-}\widetilde{\mathcal{L}}_k^{-}(x)) = (\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^{-}(x))$;
2) $\widetilde{\mathcal{L}}^{\nabla}(\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^{-}(x)) = \mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^{-}\widetilde{\mathcal{L}}_k^{-}(x)$.

According to property 7, we have obtained that $(\mathcal{H}\widetilde{\mathcal{L}}(x) \cap \mathcal{H}^{-}\widetilde{\mathcal{L}}^{-}(x), (\widetilde{\mathcal{L}}(x), \widetilde{\mathcal{L}}^{-}(x)))$ is a fuzzy three-way concept for any $x \in \Omega$. Since two pair set-valued mappings are same to the existing cognitive operators and $x \in D_i \subseteq \Omega$, then $(\mathcal{H}_k\widetilde{\mathcal{L}}_k(x) \cap \mathcal{H}_k^{-}\widetilde{\mathcal{L}}_k^{-}(x), (\widetilde{\mathcal{L}}_k(x), \widetilde{\mathcal{L}}_k^{-}(x)))$ is a F3WG-concept. #

Property 9: Let $\Omega_i^{D_k}$ be a object set about D_k under i-th cognitive state. For any object $x \in \Omega_i^{D_k}$, then the following statements hold:

1) for any $x \in \Omega_i^{D_k}$, if $x \in \Omega_{i-1}^{D_k}$, then

$$\begin{aligned}
&(\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}_{k,i}^{-}\widetilde{\mathcal{L}}_{k,i}^{-}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}_{k,i}^{-}(x)))\\
&= ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}_{k,i-1}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x)) \cup (\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\\
&\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x)), (\widetilde{\mathcal{L}}_{k,i-1}(x), \widetilde{\mathcal{L}}_{k,i-1}^{-}(x))).
\end{aligned}$$

2) for any $x \in \Omega_i^{D_k}$, if $x \notin \Omega_{i-1}^{D_k}$, then

$$\begin{aligned}
&(\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}_{k,i}^{-}\widetilde{\mathcal{L}}_{k,i}^{-}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}_{k,i}^{-}(x)))\\
&= ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x) \cap \mathcal{H}_{k,i-1}^{-}\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}(x))\\
&\cup (\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x) \cap \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}(x)),\\
&(\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x), \widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}(x))).
\end{aligned}$$

Proof: 1) If $x \in \Omega_{i-1}^{D_k}$ and $\Delta\Psi_{i-1} = \emptyset$, according to definition 6, we have $\widetilde{\mathcal{L}}_{k,i}(x) = \widetilde{\mathcal{L}}_{k,i-1}(x)$ due to $\widetilde{\mathcal{L}}_{k,\Delta\widetilde{\Psi}_{i-1}}(x) = \emptyset$ and $\widetilde{\mathcal{L}}_{k,i}^{-}(x) = \widetilde{\mathcal{L}}_{k,i-1}^{-}(x)$ due to $\widetilde{\mathcal{L}}_{k,\Delta\widetilde{\Psi}_{i-1}}^{-}(x) = \emptyset$, $\mathcal{H}_{k,i}(a) = \mathcal{H}_{k,i-1}(a) \cup \mathcal{H}_{k,\Delta\widetilde{\Omega}_{i-1}^{D_k}}(a)$ and $\mathcal{H}_{k,i}^{-}(a) = \mathcal{H}_{k,i-1}^{-}(a) \cup \mathcal{H}_{k,\Delta\widetilde{\Omega}_{i-1}^{D_k}}^{-}(a)$.

Then, according to property 6, we have

$$\begin{aligned}
&\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}_{k,i}^{-}\widetilde{\mathcal{L}}_{k,i}^{-}(x)\\
&= \mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}_{k,i}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x)\\
&= (\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x) \cup \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}_{k,i-1}(x))\\
&\cap (\mathcal{H}_{k,i-1}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x) \cup \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x))\\
&= ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x) \cup \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}_{k,i-1}(x))\\
&\cap \mathcal{H}_{k,i-1}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x))\\
&\cup ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x) \cup \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}_{k,i-1}(x))\\
&\cap \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x))\\
&= ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}_{k,i-1}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x))\\
&\cup (\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}_{k,i-1}(x)\\
&\cap \mathcal{H}_{k,i-1}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x))) \cup ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x)\\
&\cap \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x))\\
&\cup (\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}^{-}\widetilde{\mathcal{L}}_{k,i-1}^{-}(x))).
\end{aligned}$$

Furthermore, according to definition 6, we have $(\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}} \widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}^-_{k,i-1}\widetilde{\mathcal{L}}^-_{k,i-1}(x)) = \emptyset$ and $(\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}^-_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}^-_{k,i-1}(x) = \emptyset$ hold.

Thus, we conclude

$$(\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}^-_{k,i}\widetilde{\mathcal{L}}^-_{k,i}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}^-_{k,i}(x)))$$
$$= ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}^-_{k,i-1}\widetilde{\mathcal{L}}^-_{k,i-1}(x)) \cup (\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}$$
$$\widetilde{\mathcal{L}}_{k,i-1}(x) \cap \mathcal{H}^-_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}^-_{k,i-1}(x)), (\widetilde{\mathcal{L}}_{k,i-1}(x), \widetilde{\mathcal{L}}^-_{k,i-1}(x))).$$

2) If $x \notin \Omega_{i-1}^{D_k}$ and $\Delta\Psi_{i-1} = \emptyset$, according to definition 6, we have $\widetilde{\mathcal{L}}_{k,i-1}(x) = \widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x)$ due to $\widetilde{\mathcal{L}}_{k,\Delta\widetilde{\Psi}_{i-1}}(x) = \emptyset$ and $\widetilde{\mathcal{L}}^-_{k,i-1}(x) = \widetilde{\mathcal{L}}^-_{k,\Delta\Omega_{i-1}^{D_k}}(x)$ due to $\widetilde{\mathcal{L}}^-_{k,\Delta\widetilde{\Psi}_{i-1}}(x) = \emptyset$, $\mathcal{H}_{k,i}(a) = \mathcal{H}_{k,i-1}(a) \cup \mathcal{H}_{k,\Delta\widetilde{\Omega}_{i-1}^{D_k}}(a)$ and $\mathcal{H}^-_{k,i}(a) = \mathcal{H}^-_{k,i-1}(a) \cup \mathcal{H}^-_{k,\Delta\widetilde{\Omega}_{i-1}^{D_k}}(a)$.

At the same time, similar to the proof of item 1), we conclude

$$(\mathcal{H}_{k,i}\widetilde{\mathcal{L}}_{k,i}(x) \cap \mathcal{H}^-_{k,i}\widetilde{\mathcal{L}}^-_{k,i}(x), (\widetilde{\mathcal{L}}_{k,i}(x), \widetilde{\mathcal{L}}^-_{k,i}(x)))$$
$$= ((\mathcal{H}_{k,i-1}\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x) \cap \mathcal{H}^-_{k,i-1}\widetilde{\mathcal{L}}^-_{k,\Delta\Omega_{i-1}^{D_k}}(x)) \cup (\mathcal{H}_{k,\Delta\Omega_{i-1}^{D_k}}$$
$$\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x) \cap \mathcal{H}^-_{k,\Delta\Omega_{i-1}^{D_k}}\widetilde{\mathcal{L}}^-_{k,\Delta\Omega_{i-1}^{D_k}}(x)),$$
$$(\widetilde{\mathcal{L}}_{k,\Delta\Omega_{i-1}^{D_k}}(x), \widetilde{\mathcal{L}}^-_{k,\Delta\Omega_{i-1}^{D_k}}(x))).$$

By combining 1) and 2), this property is proven. #

Property 10: Let $\Omega_i^{D_k}$ be a object set about D_k under i-th cognitive state, the $\widetilde{\mathcal{BG}}_i^{D_k}$ is a fuzzy big concept subspace about the F3WG-concept subspace $\widetilde{\mathcal{G}}_i^{D_k}$. Then, the following inequality holds.

$$1 \leq |\widetilde{\mathcal{BG}}_i^{D_k}| \leq |\widetilde{\mathcal{G}}_i^{D_k}|.$$

Proof: To prove this property, we divide it into three steps:

1) For any $(X, (\widetilde{A}_1, \widetilde{A}_2)) \in \widetilde{\mathcal{G}}_i^{D_k}$, if there exist a F3WG-concept $(X', (\widetilde{A}'_1, \widetilde{A}'_2)) \in \widetilde{\mathcal{G}}_i^{D_k}$ that makes $X \subseteq X'$, from definition 9, we have $|\widetilde{\mathcal{BG}}_i^{D_k}| = 1$;
2) For any $(X, (\widetilde{A_1}, \widetilde{A_2})) \in \widetilde{\mathcal{G}}_i^{D_k}$, if there exist a F3WG-concept $(X', (\widetilde{A}', \widetilde{A}')) \in \widetilde{\mathcal{G}}_i^{D_k}$ that makes $X \nsubseteq X'$, from definition 9, we have $|\widetilde{\mathcal{BG}}_i^{D_k}| > 1$
3) For any $(X, (\widetilde{A_1}, \widetilde{A_2})) \in \widetilde{\mathcal{G}}_i^{D_k}$, if there does not exist a F3WG-concept $(X'_i, (\widetilde{A'_1}, \widetilde{A'_2})) \in \widetilde{\mathcal{S}}_i^{D_k}$ that makes $X \subset X'$, from definition 9, we have $|\widetilde{\mathcal{BG}}_i^{D_k}| \leq |\widetilde{\mathcal{G}}_i^{D_k}|$;

By combining 1), 2) and 3), this property is proven. #

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