This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems

A Regret-Based Three-Way Decision Model Under Interval Type-2 Fuzzy Environment

Tianxing Wang, Huaxiong Li, Yuhua Qian, Bing Huang, Xianzhong Zhou

***Abstract*—Three-way decision provides a new perspective for dealing with uncertainty and complexity in decision-making problems. However, behaviors of decision-makers may be influenced by different risk attitudes in reality. To address this problem, we construct a regret-based three-way decision model under interval type-2 fuzzy environment. Basically, regret theory and interval type-2 fuzzy set are utilized to improve three-way decision in coping with the risk and uncertainty. Two core issues focus on the determination of decision rules and estimation of conditional probabilities for different decision-makers under interval type-2 fuzzy environment. The maximum-utility decision rules are derived based on regret theory. An interval type-2 fuzzy technique for order preference by similarity to ideal solution (TOPSIS) method is utilized to estimate the conditional probability. The results of the illustrative example show that the proposed model can effectively solve uncertain decision problems. The comparative analysis and experimental evaluations are utilized to elaborate on the performance of the regret-based three-way decision model.**

***Index Terms*—Three-way decision, regret theory, interval type-2 fuzzy set, uncertain decision-making.**

I. Introduction

THREE-WAY decision (3WD), as a novel methodology to deal with imprecise and uncertain data, has developed rapidly in recent years. It has received extensive attention since it was proposed by Yao [1]. The idea of three-way decision originated from rough set, which also constituted a prominent knowledge discovery method for uncertain and incomplete issues [2]–[13]. The key semantic interpretation of three-way decision is to divide the whole set of objects into three disjoint regions. With respect to three regions, three-way decision provides decision-makers with three decision actions in the form of acceptance, non-commitment and rejection, respectively. The research results of three-way decision are fruitful and have been applied into numerous practical application fields [14]–[17]. For example, Li *et al.* [18] utilized a sequential three-way decision method for cost-sensitive face recognition. Jia *et al.* [19] developed a feature fusion method for Chinese irony detection in microblog based on three-way decision model. Zhan *et al.* [20] incorporated 3WD into multi-attribute decision-making based on an outranking relation and solved a problem of enterprise project investment target selections. Campagner *et al.* [21] introduced a framework based on 3WD and the trisecting-acting-outcome model to handle uncertainty in machine learning. In summary, the studies of three-way decision mainly consist of two aspects.

- Several scholars concentrated on the expansion of the reasonable semantic interpretation of the traditional three-way decision model [22]–[24]. As an example, Yao [25] introduced a model of three-way conflict analysis based on the philosophy, methodology, and mechanism in threes. Yue *et al.* [26] constructed shadowed neighborhoods based on fuzzy rough transformation for three-way classification. Jia and Liu [27] proposed a novel decision model based on the combination of three-way decisions and multi-criteria decision-making.
- Some researches focused on the development of sequential three-way decision and granular computing [28]–[30]. For instance, Li *et al.* [31] proposed a sequential granular feature extraction method based on deep neural networks. Liu *et al.* [32] discussed the methodology of 3WD via granular computing with "multi-level" strategy and "multi-view" strategy. Zhang *et al.* [33] developed a sequential three-way decision model in autoencoder based classifications and decisions.

In the case that decision-makers cope with imprecise and uncertain decision issues through three-way decision, they may be confronted with various extents of risk and uncertainty. In the practical environment, decision-makers are not entirely rational and take different decision behaviors and risk preferences. To address this problem, researchers have introduced several scientific behavioral decision theories into three-way decision, such as prospect theory and cumulative prospect theory [34]–[37]. In recent years, regret theory, proposed by Bell [38], Loomes and Sugden [39], has received sustained attention in many risk decision-making areas. It has been noted in some literature that regret theory has some advantages over prospect theory and cumulative prospect theory [40]. As an example, when using regret theory, decision-makers do not need to provide the reference point and involved parameters are fewer and simpler. Nowadays, regret theory has been successfully applied in diverse areas and achieved good results [40], [41]. For example, Peng *et al.* [42] established an applicable decision support model based on regret theory to

This work was supported by the National Natural Science Foundation of China (71671086; 71732003; 61876079), the National Key Research and Development Program of China (2018YFB1402600), and the program B for Outstanding PhD candidate of Nanjing University. *(Corresponding author: Huaxiong Li.)*

T. Wang, H. Li, and X. Zhou are with the Department of Control and Systems Engineering, Nanjing University, Nanjing, 210093, China (e-mail: wangtianxing@smail.nju.edu.cn, {huaxiongli, zhouxz}@nju.edu.cn).

Y. Qian is with the Institute of Big Data Science and Industry, Shanxi University, Taiyuan, 030006, China, and also with the Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Shanxi University, Taiyuan, 030006, China (e-mail: jinchengqyh@126.com).

B. Huang is with the School of Information Engineering, Nanjing Audit University, Nanjing, 211815, China (e-mail: hbhuangbing@126.com).

address new energy investment risk evaluation problems. Wang [43] proposed a regret-based automated Bayesian sequential decision-making strategy for the optimal allocation of manual and autonomous sensing modes. Thus, we introduce regret theory into three-way decision to describe decision-makers' risk attitudes.

Although three-way decision is a valid tool to handle decision problems under risks, it still requires to improve the capacity to process the vagueness [34]. Currently, many researches have focused on the applications with type-1 fuzzy set and achieved good performance when facing the actual decision problems [44]–[48]. Inspired by these observations, various scholars have extended three-way decision into uncertain circumstance with type-1 fuzzy set due to the constant change of fuzziness and complexity [41], [49]–[51]. As an example, Lang *et al.* [52] contributed to three-way conflict analysis based on Pythagorean fuzzy set theory. Liang *et al.* [53] combined the hesitant fuzzy information system and loss functions together via error analysis. However, some researches have indicated that interval type-2 fuzzy set (IT2FS) can achieve good performance and advantages when solving problems in uncertain and fuzzy environments [34], [54], [55]. Typically, IT2FS is regarded as an expanding format of interval type-1 fuzzy set and has been applied to dispose the ambiguity of many actual decision-making problems [56]–[58]. For instance, Dalman and Bayram [59] presented an interactive fuzzy goal programming approach for solving multiobjective nonlinear programming problems with interval type-2 fuzzy numbers (IT2FNs). Eyoh *et al.* [60] presented an approach to prediction based on a new interval type-2 Atanassov intuitionistic fuzzy logic system. Therefore, we construct the regret-based 3WD model with IT2FS to enhance its ability to deal with vagueness and uncertainty.

In three-way decision, another crucial problem is how to estimate and evaluate the conditional probability. In the majority of three-way decision models, the conditional probability is calculated by utilizing the equivalence class and information table, where one of the prerequisites is the decision attribute [1], [28], [61]. However, we may confront a universal circumstance in the actual decision environment that the information system does not have the class label or the decision attribute. For example, in many multi-attribute decision-making problems, there are only conditional attributes involved but no decision attribute in the information table [51], [62], [63]. To solve the problem in this case, Liang *et al.* [61] effectively utilized the technique for order preference by similarity to ideal solution (TOPSIS) method to determine the conditional probability in the Pythagorean fuzzy information system. Under interval type-2 fuzzy environment, many researches have extended TOPSIS method with IT2FS and achieved good performance and results [63], [64]. Inspired by these observations and successes, we effectively utilize the interval type-2 fuzzy TOPSIS method to figure out the conditional probability in the interval type-2 fuzzy information system. Liu *et al.* [65] indicated that the psychological factors of the decision-makers should be taken into account in the multi-attribute decision-making problem. Thus, we also introduce regret theory into TOPSIS method to describe decision-makers' psychological risk attitudes and calculate the conditional probability.

In order to depict decision-makers' risk attitudes and improve the capability of three-way decision to process the ambiguity and vagueness, we introduce regret theory and interval type-2 fuzzy set into three-way decision in this paper. On the whole, we sum up the mainly accomplished researches of our work. A regret-based three-way decision model is proposed with the trapezoidal IT2FNs. The interval type-2 fuzzy regret-based utility functions are calculated based on the interval type-2 fuzzy outcome matrix. The maximum-utility decision rules are determined by ranking the interval type-2 fuzzy expected utilities. Then, as the conditional probability plays an important role in three-way decision, we utilize the interval type-2 fuzzy TOPSIS method to estimate the conditional probability based on regret theory. Meanwhile, the whole decision-making process for deriving three-way decision rules is presented.

The remainder of the paper is set out as follows. Section II reviews several basic concepts of three-way decision, interval type-2 fuzzy sets and regret theory. Section III constructs the regret-based 3WD model in interval type-2 fuzzy environment. Section IV utilizes an interval type-2 fuzzy TOPSIS method to estimate the conditional probability. The illustrative example of an investment assessment problem is given to elaborate the effectiveness in Section V. Sections VI and VII implements the comparative analysis and experimental evaluations to verify the performance of the proposed model. Section VIII presents the conclusion and discusses future studies.

II. Preliminaries

This section consists of three subsections to concisely retrospect varieties of notions and models of three-way decision [1], interval type-2 fuzzy set [54], [62]–[64], [66] and regret theory [38]–[40], [67].

A. Three-way decision

Yao [1] proposed the classical three-way decision model with the aid of Bayesian decision procedure. The classical 3WD model consists of two states and three actions. A set of states $\Omega = \{C, \neg C\}$ indicates the object x is in a decision class C and not in class C. Let $\mathscr{A} = \{\pi_P, \pi_B, \pi_N\}$ be a set of actions. π_P, π_B and π_N denote the acceptance, deferment and rejection decisions respectively in classifying x into the positive region $\mathrm{POS}(C)$, boundary region $\mathrm{BND}(C)$ and negative region $\mathrm{NEG}(C)$. Let λ_{ij} $(i = P, B, N;\ j = P, N)$ denote the cost of taking actions π_P, π_B, π_N respectively when x is in a decision class C and not in class C. Thus, the cost matrix can be given by Table I. In Table I, $\lambda_{PP}, \lambda_{BP}, \lambda_{NP}$ denote the costs of taking actions π_P, π_B and π_N when x belongs to C respectively. $\lambda_{PN}, \lambda_{BN}, \lambda_{NN}$ denote the costs of taking π_P, π_B and π_N when x belongs to $\neg C$. Let us represent $Pr(C|x)$ as the conditional probability of x belongs to C. Then, the expected cost $R(\pi_i|x)$ for each object x is calculated by the following formula:

$$\begin{aligned} R(\pi_P|x) &= \lambda_{PP} Pr(C|x) + \lambda_{PN} Pr(\neg C|x), \\ R(\pi_B|x) &= \lambda_{BP} Pr(C|x) + \lambda_{BN} Pr(\neg C|x), \\ R(\pi_N|x) &= \lambda_{NP} Pr(C|x) + \lambda_{NN} Pr(\neg C|x). \end{aligned} \quad (1)$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems

TABLE I
THE COST MATRIX FOR CLASSICAL 3WD MODEL

	C	$\neg C$
π_P	λ_{PP}	λ_{PN}
π_B	λ_{BP}	λ_{BN}
π_N	λ_{NP}	λ_{NN}

By Bayesian decision procedure, the minimum-cost decision rules can be induced as:

(P0) If $R(\pi_P|x) \leq R(\pi_B|x)$ and $R(\pi_P|x) \leq R(\pi_N|x)$, decide $x \in \text{POS}(C)$;

(B0) If $R(\pi_B|x) \leq R(\pi_P|x)$ and $R(\pi_B|x) \leq R(\pi_N|x)$, decide $x \in \text{BND}(C)$;

(N0) If $R(\pi_N|x) \leq R(\pi_P|x)$ and $R(\pi_N|x) \leq R(\pi_B|x)$, decide $x \in \text{NEG}(C)$.

According to decision rules (P0) $-$ (N0), the action with the minimum expected cost is chosen as the best action.

B. Interval type-2 fuzzy set

Definition 1: [54] In the universe of discourse X, a type-2 fuzzy set $\widetilde{A}$ can be represented by a type-2 membership function $\mu_{\widetilde{A}}(x,u)$ as follows:

$$\widetilde{A} = \{((x,u), \mu_{\widetilde{A}}(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\}, \quad (2)$$

where $\mu_{\widetilde{A}}(x,u)$ denotes the membership function of $\widetilde{A}$ and satisfies: $0 \leq \mu_{\widetilde{A}}(x,u) \leq 1$. In this situation, $\widetilde{A}$ can also be represented as:

$$\widetilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\widetilde{A}}(x,u)/(x,u), \quad (3)$$

where $\int\int$ denotes union over all admissible x and u, and J_x is the primary membership of x.

Definition 2: [54] Let $\widetilde{A}$ be a type-2 fuzzy set in the universe of discourse X. If all $\mu_{\widetilde{A}}(x,u) = 1$, then $\widetilde{A}$ is called an IT2FS, which is expressed as:

$$\widetilde{A} = \int_{x \in X} \int_{u \in J_x} 1/(x,u). \quad (4)$$

In the cause of the simplification and expression of the study in this article, we utilize the trapezoidal interval type-2 fuzzy number proposed in [68]. The definition of the trapezoidal IT2FN is expressed as follows:

Definition 3: [68] Let $\widetilde{A} = (A^+, A^-)$ be a trapezoidal IT2FN in the universe of discourse, as shown in Fig. 1, where A^+ and A^- are two generalized trapezoidal fuzzy numbers:

$$\begin{aligned}\widetilde{A} &= (A^+, A^-) \\ &= ((a^+, b^+, c^+, d^+; s^+, t^+), (a^-, b^-, c^-, d^-; s^-, t^-)),\end{aligned} \quad (5)$$

where $a^+ \leq b^+ \leq c^+ \leq d^+$, $a^- \leq b^- \leq c^- \leq d^-$, $0 \leq s^- \leq s^+ \leq 1$, $0 \leq t^- \leq t^+ \leq 1$. s^+ and t^+ denote the membership values of elements b^+ and c^+. s^- and t^- denote the membership values of elements b^- and c^-.

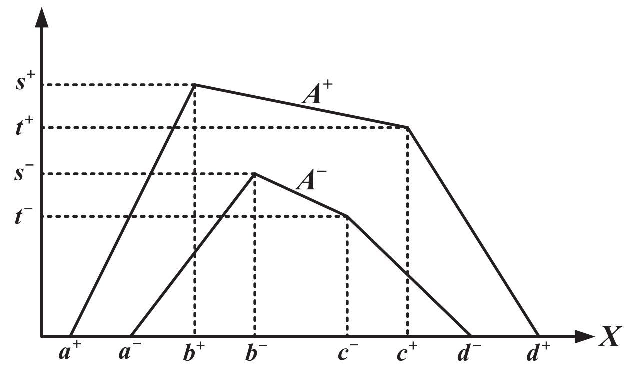


Fig. 1. The trapezoidal interval type-2 fuzzy number $\widetilde{A}$.

Definition 4: [68] Let $\widetilde{A}_1 = (A_1^+, A_1^-)$ and $\widetilde{A}_2 = (A_2^+, A_2^-)$ be two trapezoidal IT2FNs, where $\widetilde{A}_1$ and $\widetilde{A}_2$ are written as:

$$\begin{aligned}\widetilde{A}_1 &= (A_1^+, A_1^-) \\ &= ((a_1^+, b_1^+, c_1^+, d_1^+; s_1^+, t_1^+), (a_1^-, b_1^-, c_1^-, d_1^-; s_1^-, t_1^-)), \\ \widetilde{A}_2 &= (A_2^+, A_2^-) \\ &= ((a_2^+, b_2^+, c_2^+, d_2^+; s_2^+, t_2^+), (a_2^-, b_2^-, c_2^-, d_2^-; s_2^-, t_2^-)).\end{aligned} \quad (6)$$

Then, the arithmetic operations of two trapezoidal IT2FNs $\widetilde{A}_1$ and $\widetilde{A}_2$ are represented as follows:

(1) Addition operation

$$\begin{aligned}&\widetilde{A}_1 \oplus \widetilde{A}_2 = (A_1^+, A_1^-) \oplus (A_2^+, A_2^-) \\ &= \begin{pmatrix} (a_1^+ + a_2^+, b_1^+ + b_2^+, c_1^+ + c_2^+, d_1^+ + d_2^+; \\ \min\{s_1^+, s_2^+\}, \min\{t_1^+, t_2^+\}), \\ (a_1^- + a_2^-, b_1^- + b_2^-, c_1^- + c_2^-, d_1^- + d_2^-; \\ \min\{s_1^-, s_2^-\}, \min\{t_1^-, t_2^-\}) \end{pmatrix}.\end{aligned} \quad (7)$$

(2) Subtraction operation

$$\begin{aligned}&\widetilde{A}_1 \ominus \widetilde{A}_2 = (A_1^+, A_1^-) \ominus (A_2^+, A_2^-) \\ &= \begin{pmatrix} (a_1^+ - d_2^+, b_1^+ - c_2^+, c_1^+ - b_2^+, d_1^+ - a_2^+; \\ \min\{s_1^+, s_2^+\}, \min\{t_1^+, t_2^+\}), \\ (a_1^- - d_2^-, b_1^- - c_2^-, c_1^- - b_2^-, d_1^- - a_2^-; \\ \min\{s_1^-, s_2^-\}, \min\{t_1^-, t_2^-\}) \end{pmatrix}.\end{aligned} \quad (8)$$

(3) Multiplication operation

$$\begin{aligned}&\widetilde{A}_1 \otimes \widetilde{A}_2 = (A_1^+, A_1^-) \otimes (A_2^+, A_2^-) \\ &= \begin{pmatrix} (a_1^+ \times a_2^+, b_1^+ \times b_2^+, c_1^+ \times c_2^+, d_1^+ \times d_2^+; \\ \min\{s_1^+, s_2^+\}, \min\{t_1^+, t_2^+\}), \\ (a_1^- \times a_2^-, b_1^- \times b_2^-, c_1^- \times c_2^-, d_1^- \times d_2^-; \\ \min\{s_1^-, s_2^-\}, \min\{t_1^-, t_2^-\}) \end{pmatrix}.\end{aligned} \quad (9)$$

(4) Multiplication by real number operation

$$\begin{aligned}&\text{If } k \geq 0, k\widetilde{A}_1 = \begin{pmatrix} (ka_1^+, kb_1^+, kc_1^+, kd_1^+; s_1^+, t_1^+), \\ (ka_1^-, kb_1^-, kc_1^-, kd_1^-; s_1^-, t_1^-) \end{pmatrix}; \\ &\text{If } k < 0, k\widetilde{A}_1 = \begin{pmatrix} (kd_1^+, kc_1^+, kb_1^+, ka_1^+; s_1^+, t_1^+), \\ (kd_1^-, kc_1^-, kb_1^-, ka_1^-; s_1^-, t_1^-) \end{pmatrix}.\end{aligned} \quad (10)$$

TABLE II
LINGUISTIC TERMS AND THEIR CORRESPONDING INTERVAL TYPE-2 FUZZY SETS

Linguistic terms	Interval type-2 fuzzy sets
Very Poor (VP)	$((0, 0, 0, 1; 1, 1), (0, 0, 0, 0.5; 0.9, 0.9))$
Poor (P)	$((0, 1, 1, 3; 1, 1), (0.5, 1, 1, 2; 0.9, 0.9))$
Medium Poor (MP)	$((1, 3, 3, 5; 1, 1), (2, 3, 3, 4; 0.9, 0.9))$
Medium (M)	$((3, 5, 5, 7; 1, 1), (4, 5, 5, 6; 0.9, 0.9))$
Medium Good (MG)	$((5, 7, 7, 9; 1, 1), (6, 7, 7, 8; 0.9, 0.9))$
Good (G)	$((7, 9, 9, 10; 1, 1), (8, 9, 9, 9.5; 0.9, 0.9))$
Very Good (VG)	$((9, 10, 10, 10; 1, 1), (9.5, 10, 10, 10; 0.9, 0.9))$

In [66], Hu *et al.* proposed a method to rank the trapezoidal IT2FNs by calculating the ranking values. The definition is presented as follows:

Definition 5: [66] Let $\widetilde{A}$ be a trapezoidal IT2FN. The ranking value of $\widetilde{A}$ can be defined using the following form:

$$\eta(\widetilde{A}) = \frac{\sum_{l\in\{a,b,c,d\}}\sum_{\bullet\in\{+,-\}} l^{\bullet}}{8} \times \frac{\sum_{h\in\{s,t\}}\sum_{\bullet\in\{+,-\}} h^{\bullet}}{4}. \tag{11}$$

Assuming that $\widetilde{A}_1$ and $\widetilde{A}_2$ are two trapezoidal IT2FNs. Then, we get $\widetilde{A}_1 \succ \widetilde{A}_2$ if and only if $\eta(\widetilde{A}_1) > \eta(\widetilde{A}_2)$.

In many researches of interval type-2 fuzzy sets [62]–[64], interval type-2 fuzzy sets are represented and evaluated by using linguistic terms provided by experts, since the linguistic evaluation is more practical and better described quantitatively. Therefore, all the interval type-2 fuzzy variables are assessed by linguistic terms in this paper. For clarity, the linguistic terms and their corresponding interval type-2 fuzzy sets are presented in Table II. For Table II, seven linguistic terms "Very Poor (VP)", "Poor (P)", "Medium Poor (MP)", "Medium (M)", "Medium Good (MG)" , "Good (G)", "Very Good (VG)" and their corresponding interval type-2 fuzzy sets are given, respectively.

C. Regret theory

Regret theory was initially proposed by Bell [38], Loomes and Sugden [39], respectively. In the decision process, they believe that decision-makers will compare the results of their own choices with those of the other options. If decision-makers find that they can achieve better results by choosing another option, they will regret. On the contrary, they will rejoice. Therefore, the decision-maker may anticipate the regret or joy of the options during the decision process and try to avoid the choice that they will regret.

A decision-maker's comparison of two choices according to regret theory is expressed as follows. Let z_1 and z_2 be the outcomes obtained after selecting choices π_1 and π_2, respectively. The decision-maker's perceived utility for the choice π_1 is

$$v_1 = u(z_1) + r(u(z_1) - u(z_2)), \tag{12}$$

where $u(z_i)$ denotes the utility function when taking the option π_i. Based on [42], the utility function $u(z_i)$ is given as

$$u(z_i) = \frac{1 - e^{-\theta z_i}}{\theta}, \tag{13}$$

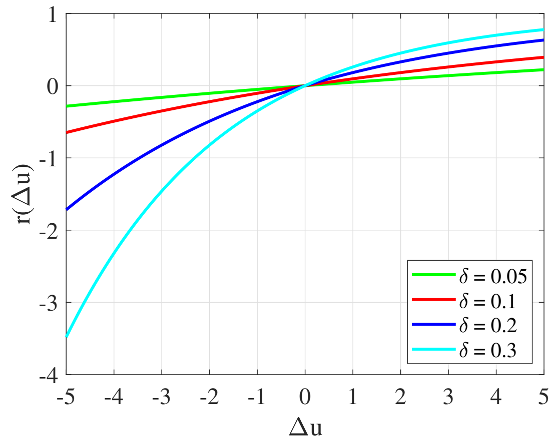


Fig. 2. The effect of δ on the regret-rejoice function $r(\Delta u)$.

where θ is the risk aversion coefficient of the decision-maker and satisfies: $0 < \theta < 1$. For (13), the utility function $u(z_i)$ will change with the variation of z_i and is monotonically increasing with the increase of z_i.

With respect to (12), the regret-rejoice function $r(\Delta u)$ measures the regret or rejoice by comparing the differences in utility between choosing π_1 and π_2. Δu denotes the difference in the utility value of two different choices. In [42], Peng *et al.* gave a form of the regret-rejoice function $r(\Delta u)$, which was defined as follows:

$$r(\Delta u) = 1 - e^{-\delta \Delta u}, \tag{14}$$

where δ denotes the regret aversion coefficient and is in the range $[0, +\infty)$.

For the regret-rejoice function $r(\Delta u)$ in (14), Fig. 2 shows the influence of δ on the regret-rejoice function $r(\Delta u)$. The regret-rejoice function $r(\Delta u)$ is also a monotonically increasing function with the increase of Δu. According to regret theory [38], [39], when $r(\Delta u) > 0$, $r(\Delta u)$ denotes the rejoice value; when $r(\Delta u) < 0$, $r(\Delta u)$ represents the regret value.

Considering the situation that the decision problem contains various options in the real world, Quiggin [67] expanded the application range of regret theory into multiple action sets to choose the optimal alternative. The decision-maker's perceived utility for the choice π_i is defined as:

$$v_i = u(z_i) + r(u(z_i) - u(z^*)), \tag{15}$$

where $z^* = \max\{z_i\}$. $r(u(z_i) - u(z^*))$ denotes the regret value, which is always non-positive.

III. THREE-WAY DECISION BASED ON REGRET THEORY

As is mentioned in Section II, the classical three-way decision model deduced the minimum-cost decision rules with the aid of Bayesian decision procedure and cost matrix. However, in the practical decision process, we need to consider the influence of different attitudes for the risk. In this section, we focus on the description on decision-makers' risk attitudes and preferences through the three-way decision model based on regret theory with trapezoidal interval type-2 fuzzy numbers.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems

TABLE III
THE INTERVAL TYPE-2 FUZZY OUTCOME MATRIX

	C	$\neg C$
π_P	$\widetilde{z}_{PP}=(z^+_{PP},z^-_{PP})$	$\widetilde{z}_{PN}=(z^+_{PN},z^-_{PN})$
π_B	$\widetilde{z}_{BP}=(z^+_{BP},z^-_{BP})$	$\widetilde{z}_{BN}=(z^+_{BN},z^-_{BN})$
π_N	$\widetilde{z}_{NP}=(z^+_{NP},z^-_{NP})$	$\widetilde{z}_{NN}=(z^+_{NN},z^-_{NN})$

TABLE IV
THE INTERVAL TYPE-2 FUZZY REGRET-BASED UTILITY FUNCTION MATRIX

	C	$\neg C$
π_P	$\widetilde{v}^k_{PP}=(v^{k,+}_{PP},v^{k,-}_{PP})$	$\widetilde{v}^k_{PN}=(v^{k,+}_{PN},v^{k,-}_{PN})$
π_B	$\widetilde{v}^k_{BP}=(v^{k,+}_{BP},v^{k,-}_{BP})$	$\widetilde{v}^k_{BN}=(v^{k,+}_{BN},v^{k,-}_{BN})$
π_N	$\widetilde{v}^k_{NP}=(v^{k,+}_{NP},v^{k,-}_{NP})$	$\widetilde{v}^k_{NN}=(v^{k,+}_{NN},v^{k,-}_{NN})$

The regret-based three-way decision model also consists of two states $\Omega=\{C,\neg C\}$ and three actions $\mathscr{A}=\{\pi_P,\pi_B,\pi_N\}$. On the basis of interval type-2 fuzzy sets and regret theory, we can take linguistic terms to evaluate the outcomes in different states when taking each action. Then, the linguistic outcomes are represented by interval type-2 fuzzy sets based on Table II. Assume that $\widetilde{z}_{ij}$ ($i=P,B,N$; $j=P,N$) is utilized to represent the interval type-2 fuzzy outcome in different states under interval type-2 fuzzy environment. All the interval type-2 fuzzy outcomes can be given by Table III. $\widetilde{z}_{PP}$, $\widetilde{z}_{BP}$, $\widetilde{z}_{NP}$ denote the interval type-2 fuzzy outcomes of taking π_P, π_B and π_N when x belongs to C respectively. $\widetilde{z}_{PN}$, $\widetilde{z}_{BN}$, $\widetilde{z}_{NN}$ denote the interval type-2 fuzzy outcomes of taking π_P, π_B and π_N when x belongs to $\neg C$. To depict the interval type-2 fuzzy outcome $\widetilde{z}_{ij}$ in detail, it is expressed as:

$$\widetilde{z}_{ij}=(z^+_{ij},z^-_{ij})=\begin{pmatrix}(a^{z,+}_{ij},b^{z,+}_{ij},c^{z,+}_{ij},d^{z,+}_{ij};s^{z,+}_{ij},t^{z,+}_{ij}),\\(a^{z,-}_{ij},b^{z,-}_{ij},c^{z,-}_{ij},d^{z,-}_{ij};s^{z,-}_{ij},t^{z,-}_{ij})\end{pmatrix}. \tag{16}$$

Suppose there are totally g decision-makers in a specific decision problem. Let $E=\{e_1,e_2,\ldots,e_g\}$ denote the decision-maker set. Regret theory illustrates that decision-makers may take different risk aversion and regret aversion coefficients. For clarity, the risk aversion coefficient of the k-th decision-maker is defined as θ_k, which satisfies: $0<\theta_k<1$. The regret aversion coefficient of the k-th decision-maker is defined as δ_k, which satisfies: $\delta_k\geq 0$. As seen in (13) and (14), the utility function and regret-rejoice function both contains the crisp value and the $\exp$ function. Thus, we give the definition of the crisp value and $\exp$ operation of trapezoidal IT2FNs.

Definition 6: Let v be a crisp value. The trapezoidal interval type-2 fuzzy representation of the crisp value v is defined as $((v,v,v,v;1,1),(v,v,v,v;1,1))$.

Definition 7: Let $\widetilde{A}=(A^+,A^-)$ be a trapezoidal interval type-2 fuzzy number. Then, the $\exp$ operation of a trapezoidal IT2FN $\widetilde{A}$ is represented as follows:

$$e^{\widetilde{A}}=\begin{pmatrix}(e^{a^+},e^{b^+},e^{c^+},e^{d^+};s^+,t^+),\\(e^{a^-},e^{b^-},e^{c^-},e^{d^-};s^-,t^-)\end{pmatrix}. \tag{17}$$

Let us define the utility function that does not consider the regret values as the common utility function. The interval type-2 fuzzy common utility function of e_k is denoted as:

$$\begin{aligned}\widetilde{u}^k_{ij}&=(u^{k,+}_{ij},u^{k,-}_{ij})\\&=\begin{pmatrix}(a^{k,u,+}_{ij},b^{k,u,+}_{ij},c^{k,u,+}_{ij},d^{k,u,+}_{ij};s^{k,u,+}_{ij},t^{k,u,+}_{ij}),\\(a^{k,u,-}_{ij},b^{k,u,-}_{ij},c^{k,u,-}_{ij},d^{k,u,-}_{ij};s^{k,u,-}_{ij},t^{k,u,-}_{ij})\end{pmatrix}.\end{aligned} \tag{18}$$

With the interval type-2 fuzzy outcomes, the interval type-2 fuzzy common utility functions of the k-th decision-maker in different states are calculated based on (13):

$$\widetilde{u}^k_{ij}=\frac{1}{\theta_k}(1\ominus e^{-\theta_k\widetilde{z}_{ij}}). \tag{19}$$

By Definition 6, we can transform the real number 1 into a trapezoidal IT2FN: $((1,1,1,1;1,1),(1,1,1,1;1,1))$. Based on Definition 4 and (17), we further calculate the interval type-2 fuzzy common utility functions as follows.

$$\widetilde{u}^k_{ij}=\begin{pmatrix}(\frac{1-e^{-\theta_k a^{z,+}_{ij}}}{\theta_k},\frac{1-e^{-\theta_k b^{z,+}_{ij}}}{\theta_k},\\ \frac{1-e^{-\theta_k c^{z,+}_{ij}}}{\theta_k},\frac{1-e^{-\theta_k d^{z,+}_{ij}}}{\theta_k};s^{z,+}_{ij},t^{z,+}_{ij}),\\ (\frac{1-e^{-\theta_k a^{z,-}_{ij}}}{\theta_k},\frac{1-e^{-\theta_k b^{z,-}_{ij}}}{\theta_k},\\ \frac{1-e^{-\theta_k c^{z,-}_{ij}}}{\theta_k},\frac{1-e^{-\theta_k d^{z,-}_{ij}}}{\theta_k};s^{z,-}_{ij},t^{z,-}_{ij})\end{pmatrix}. \tag{20}$$

According to regret theory, decision-makers will compare the results of their own choices with those of the other options. Since there are totally three actions in $\mathscr{A}=\{\pi_P,\pi_B,\pi_N\}$, we utilize regret theory in the multiple action sets [67] to construct our model. Let us define the utility function that considers the regret values as the regret-based utility function. Similar to the interval type-2 fuzzy outcomes, the interval type-2 fuzzy regret-based utility functions can be also represented by a 3×2 matrix, as shown in Table IV. For clarity, the interval type-2 fuzzy regret-based utility function of e_k is denoted as:

$$\begin{aligned}\widetilde{v}^k_{ij}&=(v^{k,+}_{ij},v^{k,-}_{ij})\\&=\begin{pmatrix}(a^{k,v,+}_{ij},b^{k,v,+}_{ij},c^{k,v,+}_{ij},d^{k,v,+}_{ij};s^{k,v,+}_{ij},t^{k,v,+}_{ij}),\\(a^{k,v,-}_{ij},b^{k,v,-}_{ij},c^{k,v,-}_{ij},d^{k,v,-}_{ij};s^{k,v,-}_{ij},t^{k,v,-}_{ij})\end{pmatrix}.\end{aligned} \tag{21}$$

Based on (15), in order to compute the interval type-2 fuzzy regret-based utility functions, we should select the decision option with the maximum outcome to calculate the regret values. Under interval type-2 fuzzy environment, we need to calculate the ranking values of all the interval type-2 fuzzy outcomes based on (11). The computed results of the ranking values of the interval type-2 fuzzy outcomes are presented as follows:

$$\begin{aligned}\eta(\widetilde{z}_{ij})=&\frac{\sum_{l\in\{a,b,c,d\}}\sum_{\bullet\in\{+,-\}}l^{z,\bullet}_{ij}}{8}\\&\times\frac{\sum_{h\in\{s,t\}}\sum_{\bullet\in\{+,-\}}h^{z,\bullet}_{ij}}{4}.\end{aligned} \tag{22}$$

When the object x belongs to C or $\neg C$, we select the interval type-2 fuzzy outcome with the highest ranking value, which is represented as follows:

$$\widetilde{z}_{*j} = \arg\max_{\widetilde{z}_{ij}} \{\eta(\widetilde{z}_{ij})\}. \tag{23}$$

At this moment, the interval type-2 fuzzy regret-based utility function $\widetilde{v}_{ij}^k$ is calculated based on (15):

$$\widetilde{v}_{ij}^k = \widetilde{u}_{ij}^k \oplus r(\widetilde{u}_{ij}^k \ominus u(\widetilde{z}_{*j})). \tag{24}$$

Based on the algebraic operations of trapezoidal IT2FNs, we can calculate the interval type-2 fuzzy regret-based utility function $\widetilde{v}_{ij}^k$ as follows:

$$\widetilde{v}_{ij}^k = \begin{pmatrix} (a_{ij}^{k,u,+} + 1 - e^{-\delta_k(a_{ij}^{k,u,+} - d_{*j}^{k,u,+})}, \\ b_{ij}^{k,u,+} + 1 - e^{-\delta_k(b_{ij}^{k,u,+} - c_{*j}^{k,u,+})}, \\ c_{ij}^{k,u,+} + 1 - e^{-\delta_k(c_{ij}^{k,u,+} - b_{*j}^{k,u,+})}, \\ d_{ij}^{k,u,+} + 1 - e^{-\delta_k(d_{ij}^{k,u,+} - a_{*j}^{k,u,+})}; \\ \min\{s_{ij}^{k,u,+}, s_{*j}^{k,u,+}\}, \min\{t_{ij}^{k,u,+}, t_{*j}^{k,u,+}\}), \\ (a_{ij}^{k,u,-} + 1 - e^{-\delta_k(a_{ij}^{k,u,-} - d_{*j}^{k,u,-})}, \\ b_{ij}^{k,u,-} + 1 - e^{-\delta_k(b_{ij}^{k,u,-} - c_{*j}^{k,u,-})}, \\ c_{ij}^{k,u,-} + 1 - e^{-\delta_k(c_{ij}^{k,u,-} - b_{*j}^{k,u,-})}, \\ d_{ij}^{k,u,-} + 1 - e^{-\delta_k(d_{ij}^{k,u,-} - a_{*j}^{k,u,-})}; \\ \min\{s_{ij}^{k,u,-}, s_{*j}^{k,u,-}\}, \min\{t_{ij}^{k,u,-}, t_{*j}^{k,u,-}\}) \end{pmatrix}. \tag{25}$$

For each decision-maker, with interval type-2 fuzzy regret-based utility functions and the conditional probability, we can calculate the interval type-2 fuzzy expected utility $\widetilde{U}^k(\pi_i|x)$ with respect to different actions in $\mathscr{A} = \{\pi_P, \pi_B, \pi_N\}$. For an object x and the k-th decision-maker, the interval type-2 fuzzy expected utility $\widetilde{U}^k(\pi_i|x)$ is expressed by the following formula:

$$\widetilde{U}^k(\pi_i|x) = Pr(C|x)\widetilde{v}_{iP}^k \oplus Pr(\neg C|x)\widetilde{v}_{iN}^k. \tag{26}$$

With the aid of algebraic operations of trapezoidal IT2FNs, the interval type-2 fuzzy expected utility $\widetilde{U}^k(\pi_i|x)$ can be further calculated as:

$$\widetilde{U}^k(\pi_i|x) = \begin{pmatrix} (a_{iP}^{k,v,+}Pr(C|x) + a_{iN}^{k,v,+}Pr(\neg C|x), \\ b_{iP}^{k,v,+}Pr(C|x) + b_{iN}^{k,v,+}Pr(\neg C|x), \\ c_{iP}^{k,v,+}Pr(C|x) + c_{iN}^{k,v,+}Pr(\neg C|x), \\ d_{iP}^{k,v,+}Pr(C|x) + d_{iN}^{k,v,+}Pr(\neg C|x); \\ \min\{s_{iP}^{k,v,+}, s_{iN}^{k,v,+}\}, \min\{t_{iP}^{k,v,+}, t_{iN}^{k,v,+}\}), \\ (a_{iP}^{k,v,-}Pr(C|x) + a_{iN}^{k,v,-}Pr(\neg C|x), \\ b_{iP}^{k,v,-}Pr(C|x) + b_{iN}^{k,v,-}Pr(\neg C|x), \\ c_{iP}^{k,v,-}Pr(C|x) + c_{iN}^{k,v,-}Pr(\neg C|x), \\ d_{iP}^{k,v,-}Pr(C|x) + d_{iN}^{k,v,-}Pr(\neg C|x); \\ \min\{s_{iP}^{k,v,-}, s_{iN}^{k,v,-}\}, \min\{t_{iP}^{k,v,-}, t_{iN}^{k,v,-}\}) \end{pmatrix}. \tag{27}$$

Regret theory illustrates that decision-makers will select the decision option with the maximum utility. For the regret-based 3WD model, decision-makers will select the action with the maximum expected utility among three actions in $\mathscr{A} = \{\pi_P, \pi_B, \pi_N\}$. Therefore, the maximum-utility decision rules of the regret-based three-way decision model in interval type-2 fuzzy environment are expressed as follows:

(P1) If $\widetilde{U}^k(\pi_P|x) \succeq \widetilde{U}^k(\pi_B|x)$ and $\widetilde{U}^k(\pi_P|x) \succeq \widetilde{U}^k(\pi_N|x)$, decide $x \in \mathrm{POS}(C)$;

(B1) If $\widetilde{U}^k(\pi_B|x) \succeq \widetilde{U}^k(\pi_P|x)$ and $\widetilde{U}^k(\pi_B|x) \succeq \widetilde{U}^k(\pi_N|x)$, decide $x \in \mathrm{BND}(C)$;

(N1) If $\widetilde{U}^k(\pi_N|x) \succeq \widetilde{U}^k(\pi_P|x)$ and $\widetilde{U}^k(\pi_N|x) \succeq \widetilde{U}^k(\pi_B|x)$, decide $x \in \mathrm{NEG}(C)$.

For decision rules $(\mathrm{P1}) - (\mathrm{N1})$, we further rank the interval type-2 fuzzy expected utility $\widetilde{U}^k(\pi_i|x)$ with (11). The ranking value of the interval type-2 fuzzy expected utility is calculated as:

$$\eta(\widetilde{U}^k(\pi_i|x)) = \frac{\sum\limits_{l\in\{a,b,c,d\}} \sum\limits_{\bullet\in\{+,-\}} (l_{iP}^{k,v,\bullet}Pr(C|x) + l_{iN}^{k,v,\bullet}Pr(\neg C|x))}{8} \times \frac{\sum_{h\in\{s,t\}}\sum_{\bullet\in\{+,-\}} \min\{h_{iP}^{k,v,\bullet}, h_{iN}^{k,v,\bullet}\}}{4}. \tag{28}$$

Therefore, the maximum-utility decision rules $(\mathrm{P2}) - (\mathrm{N2})$ are further induced as:

(P2) If $\eta(\widetilde{U}^k(\pi_P|x)) \geq \eta(\widetilde{U}^k(\pi_B|x))$ and $\eta(\widetilde{U}^k(\pi_P|x)) \geq \eta(\widetilde{U}^k(\pi_N|x))$, decide $x \in \mathrm{POS}(C)$;

(B2) If $\eta(\widetilde{U}^k(\pi_B|x)) \geq \eta(\widetilde{U}^k(\pi_P|x))$ and $\eta(\widetilde{U}^k(\pi_B|x)) \geq \eta(\widetilde{U}^k(\pi_N|x))$, decide $x \in \mathrm{BND}(C)$;

(N2) If $\eta(\widetilde{U}^k(\pi_N|x)) \geq \eta(\widetilde{U}^k(\pi_P|x))$ and $\eta(\widetilde{U}^k(\pi_N|x)) \geq \eta(\widetilde{U}^k(\pi_B|x))$, decide $x \in \mathrm{NEG}(C)$.

IV. The interval type-2 fuzzy TOPSIS method

In Section III, we discuss the determination of maximum-utility decision rules based on regret theory in interval type-2 fuzzy environment. However, how to estimate the conditional probability has not yet been proposed. In this section, we firstly discuss the calculation of the conditional probability by the interval type-2 fuzzy TOPSIS method. Then, the key steps and the whole decision procedure of the regret-based 3WD model are emphasized.

A. The estimation of the conditional probability

In three-way decision, another crucial problem is how to estimate and evaluate the conditional probability. As shown in (26) and (27), we should firstly calculate the conditional probability $Pr(C|x)$ for each object before determining the interval type-2 fuzzy expected utility $\widetilde{U}^k(\pi_i|x)$. In most 3WD models, the conditional probability is calculated by utilizing the equivalence class and information table, where the prerequisite is the decision attribute [1], [28], [51], [61]. However, in the real world, we may confront a common information system without the class label. Inspired by [61], [63]–[65], we utilize

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems

the interval type-2 fuzzy TOPSIS method to figure out the conditional probability in the interval type-2 fuzzy information system and involve regret theory in TOPSIS method to reflect decision-makers' psychological risk attitudes and preferences.

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of objects. The attribute set of the interval type-2 fuzzy information system is expressed as $F = \{f_1, f_2, \ldots, f_n\}$. $W = \{w_1, w_2, \ldots, w_n\}^T$ is the weight vector of all attributes, where w_q is the real number and satisfies: $0 \leq w_q \leq 1$ and $\sum_{q=1}^{n} w_q = 1$. As is mentioned in Section III, θ_k and δ_k are denoted as the risk aversion and regret aversion coefficients of the decision-maker e_k respectively. Note that the interval type-2 fuzzy information system in this paper does not have the class label.

According to interval type-2 fuzzy sets in Section II, we take the linguistic terms represented by IT2FS to obtain the evaluation information under interval type-2 fuzzy environment. Thus, the evaluation results in the information system are firstly assessed by linguistic terms provided by experts, and then transformed into trapezoidal IT2FNs by Table II. For clarity, the interval type-2 fuzzy information system is illustrated in Table V, where the evaluation result $\widetilde{A}_{pq}$ is the trapezoidal IT2FN and denoted as follows:

$$\widetilde{A}_{pq} = (A^+_{pq}, A^-_{pq}) = \begin{pmatrix} (a^+_{pq}, b^+_{pq}, c^+_{pq}, d^+_{pq}; s^+_{pq}, t^+_{pq}), \\ (a^-_{pq}, b^-_{pq}, c^-_{pq}, d^-_{pq}; s^-_{pq}, t^-_{pq}) \end{pmatrix}. \quad (29)$$

The motivation of the TOPSIS approach is to pick up the alternative which takes the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. According to the risk aversion coefficient θ_k, all the evaluation results in the interval type-2 fuzzy information system should be transformed into the evaluation utilities for the decision-maker e_k based on (13) as follows:

$$u_k(\widetilde{A}_{pq}) = \frac{1}{\theta_k}(1 \ominus e^{-\theta_k \widetilde{A}_{pq}}). \quad (30)$$

Based on the arithmetic operations of trapezoidal IT2FNs, we further calculate the interval type-2 fuzzy evaluation utility $u_k(\widetilde{A}_{pq})$ as follows.

$$u_k(\widetilde{A}_{pq}) = \begin{pmatrix} (\frac{1-e^{-\theta_k a^+_{pq}}}{\theta_k}, \frac{1-e^{-\theta_k b^+_{pq}}}{\theta_k}, \\ \frac{1-e^{-\theta_k c^+_{pq}}}{\theta_k}, \frac{1-e^{-\theta_k d^+_{pq}}}{\theta_k}; s^+_{pq}, t^+_{pq}), \\ (\frac{1-e^{-\theta_k a^-_{pq}}}{\theta_k}, \frac{1-e^{-\theta_k b^-_{pq}}}{\theta_k}, \\ \frac{1-e^{-\theta_k c^-_{pq}}}{\theta_k}, \frac{1-e^{-\theta_k d^-_{pq}}}{\theta_k}; s^-_{pq}, t^-_{pq}) \end{pmatrix}. \quad (31)$$

Then, for every decision-maker e_k, we can calculate the ranking value of $u_k(\widetilde{A}_{pq})$ in the interval type-2 fuzzy information system as:

$$\begin{aligned} \eta^{u,k}_{pq} =& \eta(u_k(\widetilde{A}_{pq})) \\ =& \frac{\sum_{l\in\{a,b,c,d\}} \sum_{\bullet\in\{+,-\}} (1-e^{-\theta_k l^\bullet_{pq}})/\theta_k}{8} \\ &\times \frac{\sum_{h\in\{s,t\}} \sum_{\bullet\in\{+,-\}} h^\bullet_{pq}}{4}, \end{aligned} \quad (32)$$

TABLE V
THE INTERVAL TYPE-2 FUZZY INFORMATION SYSTEM

	f_1	f_2	...	f_n
x_1	$\widetilde{A}_{11}$	$\widetilde{A}_{12}$	...	$\widetilde{A}_{1n}$
x_2	$\widetilde{A}_{21}$	$\widetilde{A}_{22}$	...	$\widetilde{A}_{2n}$
...	...	...	...	...
x_m	$\widetilde{A}_{m1}$	$\widetilde{A}_{m2}$	...	$\widetilde{A}_{mn}$

where $1 \leq p \leq m$ and $1 \leq q \leq n$.

Afterwards, we need to determine the positive ideal solution $x^+_k = (v^{+,k}_1, v^{+,k}_2, \ldots, v^{+,k}_n)$ and the negative ideal solution $x^-_k = (v^{-,k}_1, v^{-,k}_2, \ldots, v^{-,k}_n)$, where

$$v^{+,k}_q = \max_{1\leq p\leq m} \{\eta^{u,k}_{pq}\}, \quad (33)$$

and

$$v^{-,k}_q = \min_{1\leq p\leq m} \{\eta^{u,k}_{pq}\}. \quad (34)$$

Subsequently, we should calculate the distance $d^+_k(x_p)$ between each object x_p and the positive ideal solution x^+_k, and the distance $d^-_k(x_p)$ between each object x_p and the negative ideal solution x^-_k. Motivated by regret theory, the calculation of distances may also consider the regret or rejoice values between objects and ideal solutions based on (14), which is shown as follows:

$$\begin{aligned} d^+_k(x_p) &= \sqrt{\sum_{q=1}^{n} w_q \times (\eta^{u,k}_{pq} + 1 - e^{-\delta_k(\eta^{u,k}_{pq} - v^{+,k}_q)} - v^{+,k}_q)^2}, \\ d^-_k(x_p) &= \sqrt{\sum_{q=1}^{n} w_q \times (\eta^{u,k}_{pq} + 1 - e^{-\delta_k(\eta^{u,k}_{pq} - v^{-,k}_q)} - v^{-,k}_q)^2}. \end{aligned} \quad (35)$$

Finally, the relative closeness of x_p to the positive ideal solution x^+_k can be computed as follows:

$$RC_k(x_p) = \frac{d^-_k(x_p)}{d^+_k(x_p) + d^-_k(x_p)}. \quad (36)$$

Liang *et al.* [61] indicated that the relative closeness can reflect the conditional probability of the object that belongs to C. Let us denote the conditional probability of the object x_p for the decision-maker e_k as $Pr_k(C|x_p)$. Therefore, for each decision-maker $e_k \in E$ and object $x_p \in X$, we estimate the conditional probability of the object x_p as: $Pr_k(C|x_p) = RC_k(x_p)$.

B. The whole decision procedure of regret-based three-way decision model

By concluding the aforementioned models and approaches, we outline core steps of the whole decision procedure for the regret-based 3WD model under interval type-2 fuzzy environment. The evaluation and decision process is shown in Fig. 3. For the description of the whole decision procedure of the proposed regret-based 3WD model in Fig. 3, decision-makers may select different risk aversion and regret aversion

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems

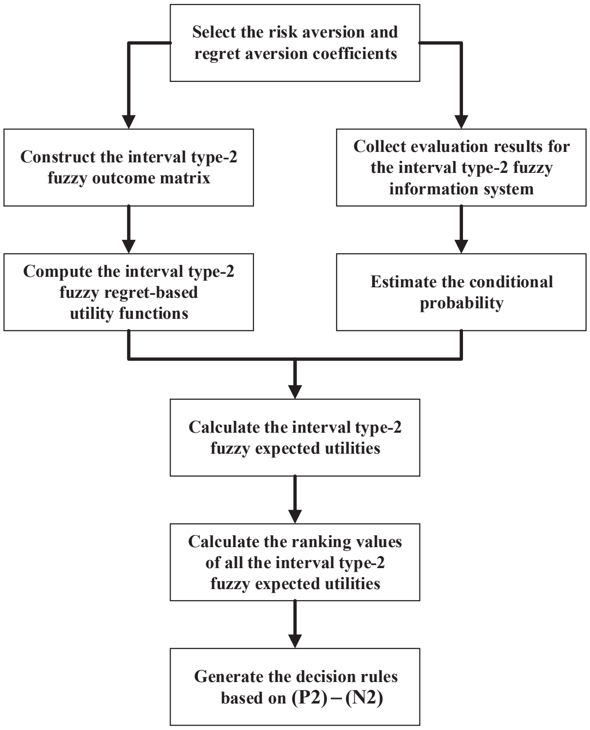


Fig. 3. The whole decision-making process of the regret-based 3WD model.

coefficients according to their own decision preferences. Then, different interval type-2 fuzzy regret-based utility functions and conditional probabilities may be obtained, which will derive diverse decision rules and results for each decision-maker. In order to better illustrate the whole decision process of the proposed model, six key steps are summarized as follows:

Step 1: For the decision-maker $e_k \in E$, select the risk aversion coefficient θ_k and the regret aversion coefficient δ_k.

Step 2: Construct the interval type-2 fuzzy outcome matrix. Then, compute the interval type-2 fuzzy regret-based utility function $\widetilde{v}_{ij}^k$.

Step 3: Collect evaluation results for the interval type-2 fuzzy information system. Subsequently, calculate the relative closeness $RC_k(x)$ for $x \in X$ and estimate the conditional probability.

Step 4: Compute the interval type-2 fuzzy expected utility $\widetilde{U}^k(\pi_i|x)$ with respect to different actions in $\mathscr{A} = \{\pi_P, \pi_B, \pi_N\}$ with (27).

Step 5: Calculate the ranking value $\eta(\widetilde{U}^k(\pi_i|x))$ of the interval type-2 fuzzy expected utility with (28).

Step 6: Compare the ranking value $\eta(\widetilde{U}^k(\pi_i|x))$ to generate the three-way decision rules with $(\mathrm{P2}) - (\mathrm{N2})$ for $x \in X$.

V. ILLUSTRATIVE EXAMPLE

In this section, we utilize the regret-based three-way decision model to solve an investment assessment problem under interval type-2 fuzzy environment.

A. Problem description

With the continuous development of the economy, China's consumption of natural resources is also increasing [69]. However, the limitation of local resources is restricting future development, so more and more companies choose overseas countries to seek investment opportunities [70]. Suppose the overseas investment department of a company decides to conduct an assessment of six candidate countries and choose the suitable countries for the investment. With respect to this investment assessment problem, assume that there are four factors of evaluation, including "Resources", "Politics and Policy", "Economy", and "Infrastructure" [66]. For six candidate countries and four factors, we obtain: $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $F = \{f_1, f_2, f_3, f_4\}$. In this investment assessment problem, suppose there are six decision-makers assessing the candidate countries, which are denoted as $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

For this investment assessment problem, we utilize the proposed regret-based 3WD model in this paper to solve the problem. For each candidate country, it has two states $\Omega = \{C, \neg C\}$ and three actions $\mathscr{A} = \{\pi_P, \pi_B, \pi_N\}$. As is mentioned in Section II, the corresponding information system and outcome matrix are represented and evaluated by using linguistic terms provided by experts. Table VI presents the linguistic outcome matrix of the investment assessment problem. The corresponding linguistic information system is presented in Table VII. In this investment assessment problem, the weight vector of all attributes is directly given as: $W = \{0.25, 0.2, 0.25, 0.3\}^T$.

TABLE VI
THE LINGUISTIC OUTCOME MATRIX OF THE ILLUSTRATIVE EXAMPLE

	C	$\neg C$
π_P	MG	P
π_B	M	MP
π_N	VP	MG

TABLE VII
THE LINGUISTIC INFORMATION SYSTEM OF THE ILLUSTRATIVE EXAMPLE

	f_1	f_2	f_3	f_4
x_1	G	G	MG	G
x_2	G	M	G	VG
x_3	VG	M	G	MG
x_4	VG	G	MG	M
x_5	VG	VG	MP	MG
x_6	G	VG	MP	M

B. Decision analysis with the proposed model

With the aid of the proposed regret-based 3WD model, we discuss the evaluation of the investment assessment problem and determine the corresponding decision steps in detail as follows:

Step 1: For all six decision-makers in E, they need to select the risk aversion coefficient θ_k and the regret aversion coefficient δ_k according to their own decision preferences. In this problem, the risk aversion and regret aversion coefficients of six decision-makers are all given in Table VIII.

TABLE VIII
THE RISK AVERSION AND REGRET AVERSION COEFFICIENTS OF SIX DECISION-MAKERS

	e_1	e_2	e_3	e_4	e_5	e_6
θ_k	0.55	0.45	0.35	0.25	0.15	0.05
δ_k	0.25	0.2	0.15	0.1	0.05	0

TABLE IX
THE DECISION RESULTS OF EACH CANDIDATE FOR EVERY DECISION-MAKER

	POS(C)	BND(C)	NEG(C)
e_1	$\{x_1\}$	$\{x_2, x_3, x_4, x_5, x_6\}$	$\emptyset$
e_2	$\{x_1\}$	$\{x_2, x_3, x_4, x_5, x_6\}$	$\emptyset$
e_3	$\{x_1\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_6\}$
e_4	$\{x_1\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_6\}$
e_5	$\{x_1, x_2\}$	$\{x_3, x_4, x_5\}$	$\{x_6\}$
e_6	$\{x_1, x_2, x_3\}$	$\{x_4\}$	$\{x_5, x_6\}$

Step 2: Based on Tables II and VI, we construct the interval type-2 fuzzy outcome $\widetilde{z}_{ij}$ of the investment assessment problem. Then, we compute the interval type-2 fuzzy regret-based utility function $\widetilde{v}_{ij}^k$ for each decision-maker according to the risk aversion and regret aversion coefficients in Table VIII.

Step 3: Based on Tables II and VII, we collect the interval type-2 fuzzy evaluation result $\widetilde{A}_{pq}$ for the interval type-2 fuzzy information system. Subsequently, the relative closeness $RC_k(x)$ for all six candidate countries is calculated and we estimate the conditional probability $Pr_k(C|x)$ for each candidate and decision-maker with the regret-based TOPSIS approach.

Step 4: By combining the interval type-2 fuzzy regret-based utility function $\widetilde{v}_{ij}^k$ and the conditional probability $Pr_k(C|x)$, we further compute the interval type-2 fuzzy expected utility $\widetilde{U}^k(\pi_i|x)$ with respect to different actions in $\mathscr{A} = \{\pi_P, \pi_B, \pi_N\}$ with (27).

Step 5: We calculate the ranking value $\eta(\widetilde{U}^k(\pi_i|x))$ of the interval type-2 fuzzy expected utility with (28) for every decision-maker.

Step 6: In light of the decision rules (P2) – (N2), we can deduce all the decision results for six candidate countries and decision-makers, which are shown in Table IX. From Table IX, the detailed results and conclusions are discussed as follows:

(1) For the decision-makers e_1 and e_2, the decision results are deduced as: POS(C) $= \{x_1\}$, BND(C) $= \{x_2, x_3, x_4, x_5, x_6\}$. The decision results imply that the decision-makers e_1 and e_2 should both accept the candidate country x_1. Meanwhile, x_2, x_3, x_4, x_5, x_6 need to be further investigated for their judgements.

(2) For the decision-makers e_3 and e_4, the decision results for six candidate countries are determined: POS(C) $= \{x_1\}$, BND(C) $= \{x_2, x_3, x_4, x_5\}$, NEG(C) $= \{x_6\}$. They indicate that the decision-makers e_3 and e_4 should accept x_1, reject x_6 and reconsider x_2, x_3, x_4, x_5.

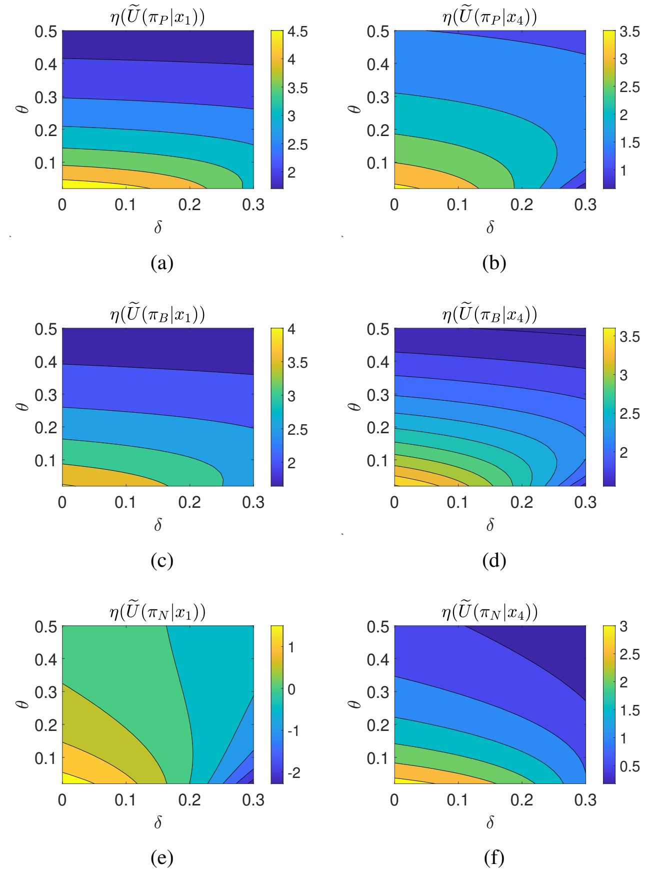


Fig. 4. Sensitivity analysis of the ranking values of the interval type-2 fuzzy expected utilities $\widetilde{U}(\pi_i|x_1)$ and $\widetilde{U}(\pi_i|x_4)$ for varying values of θ and δ.

(3) For the decision-maker e_5, we deduce the decision results as: POS(C) $= \{x_1, x_2\}$, BND(C) $= \{x_3, x_4, x_5\}$, NEG(C) $= \{x_6\}$. Therefore, the decision-maker ought to accept x_1, x_2, reconsider x_3, x_4, x_5 and reject x_6.

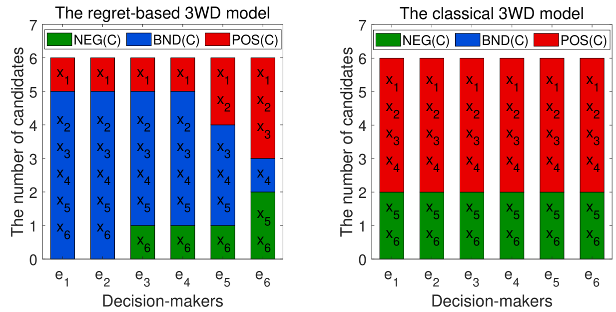
(4) For the decision-maker e_6, the decision results are expressed as: POS(C) $= \{x_1, x_2, x_3\}$, BND(C) $= \{x_4\}$, NEG(C) $= \{x_5, x_6\}$. The results show that x_1, x_2, x_3 should be accepted and x_5, x_6 will be rejected. Meanwhile, the candidate country x_4 needs to be further investigated.

C. Sensitivity analysis

With above discussion, it is essential to note that the risk aversion coefficient θ and the regret aversion coefficient δ are two fundamental components of the regret-based three-way decision model. As θ and δ are two free parameters, we can limit their values to study the change of the decision rules and results from the viewpoint of the ranking values of the interval type-2 fuzzy expected utilities. For simplicity, we only study the change rule of the objects x_1 and x_4 in this section.

For two candidate countries x_1 and x_4, we have experimented with 25 different values of θ from 0.02 to 0.50 with a step size of 0.02. The value 0.00 of θ is not taken into account since the denominator of (13) cannot be the zero. At the same time, 31 different values of δ are considered from 0.00 to 0.30

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems



(a) (b)

Fig. 5. Comparative analysis of the decision rules and results between the regret-based 3WD model and classical 3WD model.

with a step size of 0.01. Therefore, with the change of values of θ and δ, the detailed results of the ranking values of the interval type-2 fuzzy expected utilities are shown in Fig. 4. For Fig. 4, we present a kind of three-dimensional plot for varying values of θ and δ. We observe that the ranking values of the interval type-2 fuzzy expected utility may exhibit different trends for increasing values of θ and δ associated with taking each action. The research indicates that the selection of the risk aversion coefficient θ and the regret aversion coefficient δ takes great influence on decision rules and results.

VI. Comparative analysis

In this section, we make the comparative analysis to analyze the effectiveness and superiority of the proposed model.

A. Comparative analysis with the classical 3WD model

Firstly, we make the comparative analysis of our model with the classical 3WD model [1]. For fair comparison, we generalize the classical three-way decision model [1] into the interval type-2 fuzzy environment and compare it with our model, since the classical 3WD model was originally constructed with real numbers. The classical 3WD model is also utilized to deal with the above investment assessment problem. Since the classical 3WD model utilizes the cost matrix to derive decision rules, we make a transformation on the linguistic outcomes of the investment assessment problem in Table VI by transforming the index of linguistic terms in the linguistic term set for simplicity. Then, based on Table II, we can similarly obtain the interval type-2 fuzzy cost matrix for the classical 3WD model.

By using the TOPSIS method in [63], we can also determine the conditional probability of the classical 3WD model under interval type-2 fuzzy environment based on Tables II and VII. With interval type-2 fuzzy costs and conditional probabilities, we compute the interval type-2 fuzzy expected costs and their ranking values. Then, we similarly generate the minimum-cost decision rules, and deduce the decision results for six candidate countries and decision-makers. As a result, for all the six decision-makers, the decision results for six candidate countries are determined: $\mathrm{POS}(C) = \{x_1, x_2, x_3, x_4\}$, $\mathrm{NEG}(C) = \{x_5, x_6\}$. The decision results imply that all the six decision-makers will accept x_1, x_2, x_3, x_4 and reject x_5, x_6.

As the decision results of the regret-based 3WD model and the classical 3WD model are both determined, we further compare the decision rules and results between two 3WD models, as shown in Fig. 5. With regard to Fig. 5, x-coordinate denotes the decision-makers and y-coordinate pertains to the number of candidates classified in each region. For Fig. 5, we obtain that the decision rules and results of the classical 3WD model are not going to change with the variation of different decision-makers in interval type-2 fuzzy environment. On the contrary, the decision rules and results of the regret-based 3WD model are varied for different decision-makers, since the risk aversion coefficient θ_k and regret aversion coefficient δ_k are both considered in the proposed model. Besides, the results of Fig. 5 indicate that the proposed method can manage to magnify or reduce the size of the boundary region in the actual decision circumstance according to decision-makers' decision preferences.

B. Comparative analysis of the risk attitude with other two 3WD models

In [34], [41], Liang *et al.* also constructed two different three-way decision models to describe and reflect decision-makers' risk attitudes. Thus, we further make the comparative analysis and discussions between our proposed regret-based 3WD model and two 3WD models in [34], [41].

For clarity, all the similarities and differences are summed up in Table X. As shown in Table X, 3WD model in [34], 3WD model in [41] and our model have three same characteristics. The risk attitude, fuzzy environment and utility-based decision rules are all considered in three different models. This can also reflect to some extent that it is very important and reasonable to improve the capacity of three-way decision to process the risk and vagueness under uncertainty.

By comparing the 3WD model in [34] and our model, two different 3WD models also have two other same characteristics, apart from the above three similarities. Both two 3WD models utilize IT2FS to cope with fuzzy environment and describe the risk attitude from the perspective of gain and loss. However, as shown in Table X, two main contributions and advantages are obtained in our model. First, our model introduces regret theory into TOPSIS to estimate the conditional

TABLE X
THE COMPARATIVE ANALYSIS BETWEEN THREE 3WD MODELS

	Model in [34]	Model in [41]	Our model
Risk attitude	✓	✓	✓
Fuzzy environment	✓	✓	✓
Utility-based decision rules	✓	✓	✓
IT2FS	✓	×	✓
Gain and loss	✓	×	✓
Preference for $Pr(C\|x)$	×	×	✓
Parameter number	3	2	2

Note: ✓ denotes yes and × denotes no.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems

TABLE XI
THE CONDITIONAL PROBABILITY RESULTS OF THREE METHODS

	$Pr_3(C\|x_2)$	$Pr_6(C\|x_2)$	Rank of candidate countries
Method in [61]	0.6397	0.6397	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$
Method in [51]	0.5566	0.5566	$x_2 \succ x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_6$
Our method	0.7242	0.6501	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$

probability, where the preferences of different decision-makers are also involved. Second, the parameter number of our model is smaller than that of the 3WD model in [34], which is simpler in the actual circumstance.

Then, we also make the comparison and discussion between the 3WD model in [41] and our model. For Table X, three main differences between two 3WD models are included. First, IT2FS is utilized in our model, while the 3WD model in [41] considers the interval-valued environment. Second, our model describes the risk attitude from the perspective of both gain and loss, while the 3WD model in [41] derives the decision rules only from loss functions. Finally, our model introduces regret theory into TOPSIS to estimate the conditional probability, where the preferences of different decision-makers are also involved.

C. Comparative analysis of the determination of the conditional probability with other two methods

In the universal circumstance that the information system does not have the class label or the decision attribute, Liang *et al.* [61] effectively utilized TOPSIS method to determine the conditional probability in the Pythagorean fuzzy environment. Besides, Liu *et al.* [51] used grey relational degree to calculate the conditional probability with intuitionistic fuzzy numbers. In order to verify the effectiveness and superiority of the proposed regret-based TOPSIS method, we further extend two methods in [51], [61] into the interval type-2 fuzzy environment to deal with the above investment assessment problem and make the comparative analysis between three different approaches.

For the above investment assessment problem, the conditional probabilities of six candidate countries can be all calculated for six decision-makers by utilizing two methods in [51], [61]. Therefore, the results of the calculated conditional probabilities of three methods are shown in Table XI. For simplicity, we only present the rank of six candidates for six decision-makers and the detailed conditional probabilities $Pr_3(C|x_2)$ and $Pr_6(C|x_2)$ of the candidate country x_2 for two decision-makers e_3 and e_6 as the example.

From Table XI, the conditional probability results indicate that the rank result of our method is the same as that of the method in [61], and different from that of the method in [51]. This shows the difference in principle between TOPSIS method and grey relational degree. From the ranks of six candidates in Table XI, the result reflects that the candidate countries x_1, x_2, x_3 are most likely to be accepted and x_4, x_5, x_6 are most likely to be rejected for all six decision-makers and three methods, which demonstrates that our method is persuasive and reasonable. However, for $Pr_3(C|x_2)$ and $Pr_6(C|x_2)$, our method can evaluate different conditional probabilities of x_2 for e_3 and e_6, while $Pr_3(C|x_2) = Pr_6(C|x_2)$ holds for other two methods in [51], [61]. This demonstrates that our method can reflect and describe different risk attitudes and preferences of decision-makers when evaluating the conditional probability.

TABLE XII
THE DESCRIPTION OF THE EXPERIMENTAL DATA SETS

ID	Data sets	Objects	Attributes	$C : \neg C$
1	Connectionist Bench	208	61	111 : 97
2	Ionosphere	351	35	225 : 126
3	Parkinson Speech	1040	29	520 : 520
4	Wdbc	569	31	357 : 212

TABLE XIII
THE LINGUISTIC OUTCOME MATRIX OF THE EXPERIMENT

	C	$\neg C$
π_P	MG	VP
π_B	MP	MP
π_N	VP	G

VII. EXPERIMENTAL EVALUATIONS

In this section, we conduct some experiments to demonstrate the effectiveness and performance of our proposed regret-based three-way decision model. The data sets utilized in our experiments are downloaded from the machine learning data repository, University of California at Irvine (UCI) (http://archive.ics.uci.edu/ml/). The description of four experimental data sets is presented in Table XII. All the experiments are implemented by using Matlab R2019b on a personal computer with Microsoft Windows 10, Intel (R) Core (TM) i5-8265U CPU @ 1.60 GHz and 8.00 GB memory.

Compared with the two-way decision process, the advantage of three-way decision is to add the boundary region and take the non-commitment as the delayed decision. Thus, we deeply examine the influence of different values of the parameters θ and δ to the variation of three regions, since the risk aversion coefficient θ and regret aversion coefficient δ are two key parameters to our model. For four different data sets, the conditional probability of each object is calculated with the interval type-2 fuzzy TOPSIS method. Apparently, it is necessary to normalize different types of attributes into interval type-2 fuzzy sets. In this experiment, the types of attributes are first normalized into the linguistic terms, and then represented by their corresponding interval type-2 fuzzy sets in Table II. For the conditional attribute of each data set, we assume that the weight is the same. The outcome matrix in this experiment is also given by linguistic terms, which is shown in Table XIII. With decision rules (P2) $-$ (N2), results of the number of objects in three different regions for each data set are shown in Fig. 6.

For Fig. 6, we elaborately study the variation of positive, boundary and negative regions with the change of the risk

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems

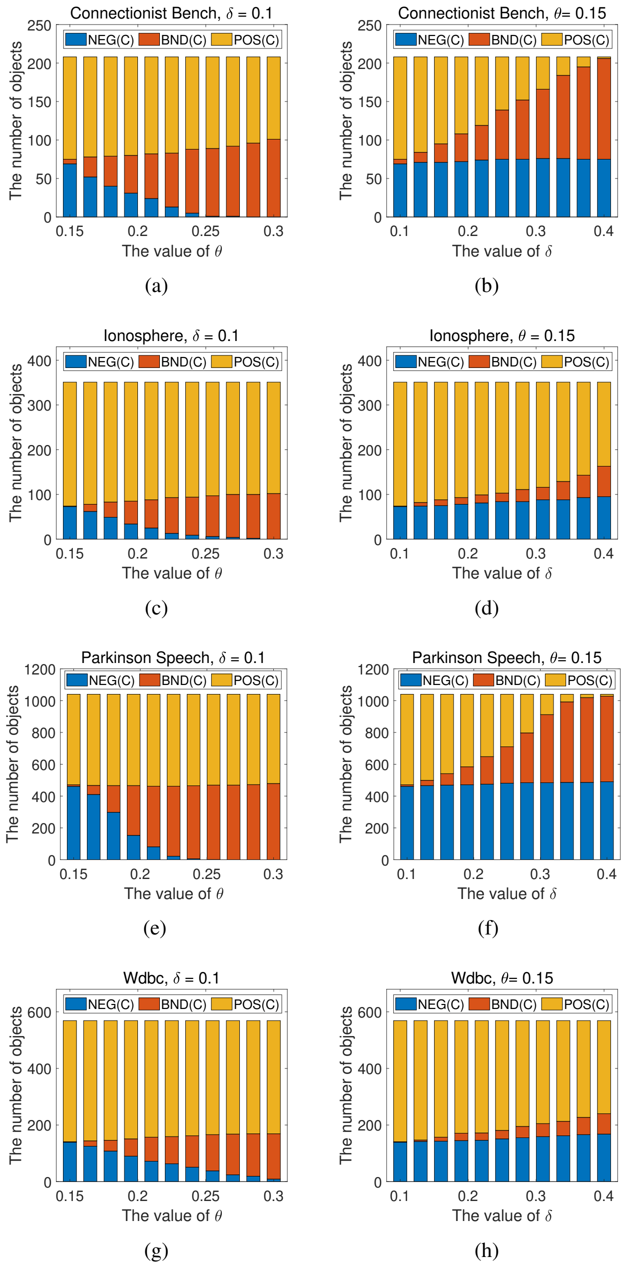


Fig. 6. The variation of objects in three regions with the change of the risk aversion coefficient θ and regret aversion coefficient δ respectively.

aversion coefficient θ and regret aversion coefficient δ, respectively. The x-coordinate represents the value of risk aversion coefficient θ or regret aversion coefficient δ. The y-coordinate concerns the number of objects. Through a detailed research of Fig. 6, some interesting points are worth discussing carefully.

(1) We examine the variation of three regions for varying values of θ when $\delta = 0.1$. At this time, we have experimented with 11 different values of θ from 0.15 to 0.3 with a step size of 0.015. From Fig. 6, we observe that the number of objects in the boundary region will become larger with the increase of θ. On the contrary, the number of objects in the positive and negative regions will become smaller with the increase of θ. The influence of θ on the change of negative and boundary regions is greater than that of the positive region.

(2) We research the variation of three regions for varying values of δ when $\theta = 0.15$. Similarly, 11 different values of δ are considered from 0.1 to 0.4 with a step size of 0.03. From Fig. 6, the results indicate that the trends and patterns of the change of the positive and boundary regions with the increase of δ are as same as those with the increase of θ. The number of objects in the negative region will not decrease with the increase of δ, which is different from that with the increase of θ. The effect of δ on the change of positive and boundary regions is greater than that of the negative region.

(3) The results of Fig. 6 also indicate that if decision-makers would rather make certainty decision, they should set values of θ and δ smaller. On the contrary, if decision-makers prefer more objects should be included in the boundary region to make delayed decision, the values of θ and δ need to set bigger. The conclusion is consistent with the decision results in Tables VIII and IX.

As the decision results of all the objects in four data sets are determined in Fig. 6 with the change of θ and δ, it is necessary to evaluate the performance of the proposed model by utilizing the decision attribute in four data sets. Inspired by [71], we utilize the error rate to evaluate the performance of our proposed regret-based 3WD model. The error rate is defined as follows [71]:

$$\text{Error rate} = \frac{n_{C \to \text{NEG}(C)} + n_{\neg C \to \text{POS}(C)}}{N} \times 100\%, \quad (37)$$

where N is the total number of objects in the universe. $n_{C \to \text{NEG}(C)}$ represents the number of objects classified into the negative region $\text{NEG}(C)$ belonging to C. $n_{\neg C \to \text{POS}(C)}$ denotes the number of objects classified into the positive region $\text{POS}(C)$ belonging to $\neg C$. Apparently, the smaller of the error rate can reflect that the performance of the proposed regret-based 3WD model is better.

Therefore, the results of the error rates for all the objects in four data sets are presented in Fig. 7. Fig. 7 (a) shows the change of the error rates for four data sets with the increase of θ when $\delta = 0.1$, where the x-coordinate represents the value of the risk aversion coefficient θ from 0.15 to 0.3, and the y-coordinate concerns the error rate from 4% to 16%. Fig. 7 (b) shows the change of the error rates for four data sets with the increase of δ when $\theta = 0.15$, where the x-coordinate represents the value of the regret aversion coefficient δ from 0.1 to 0.4, and the y-coordinate denotes the error rate from 0% to 15%.

From Fig. 7, we observe that the error rate is significantly affected by the values of θ and δ. With the increase of θ or δ, the error rate of all the four data sets is decreasing since the boundary region $\text{BND}(C)$ is magnified. This implies the advantage of three-way decision over two-way decision. In the actual decision environment, decision-makers can enlarge the boundary region and reduce the error rate by changing risk aversion and regret aversion coefficients θ and δ. This also reflects the validity of the proposed regret-based three-way

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems

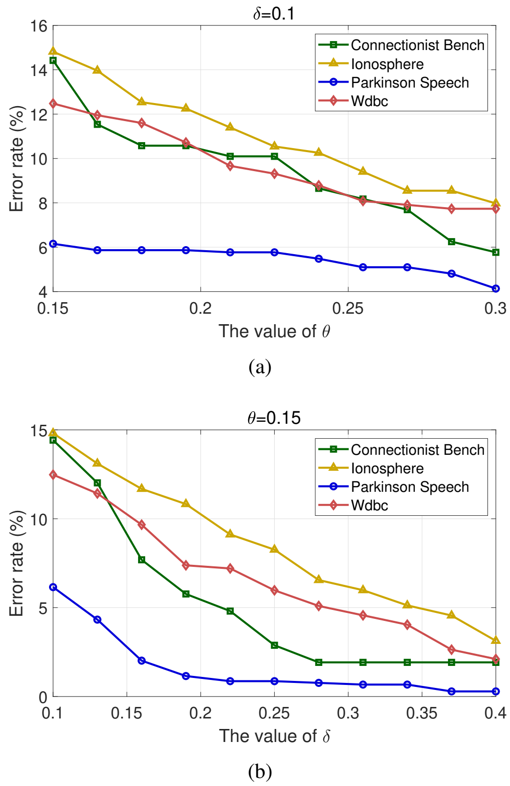


Fig. 7. The error rates of four data sets with the variation of θ and δ.

decision model in this paper and verifies its performance in interval type-2 fuzzy environment.

VIII. Conclusion

In order to describe the effect of decision-makers' risk attitudes more appropriately, this paper proposes a regret-based three-way decision model under interval type-2 fuzzy environment. For the description on the psychological risk attitudes and preferences, the regret-based 3WD model can derive different maximum-utility decision rules for decision-makers according to their risk aversion and regret aversion coefficients. The conditional probability is evaluated with the interval type-2 fuzzy TOPSIS method based on regret theory. The proposed model is verified through an illustrative example of an investment assessment problem and the comparative analysis. Finally, the results of the experimental evaluations show the effectiveness and performance of the regret-based 3WD model. In the future, we will consider the extension of the proposed model in the multi-attribute decision-making problem and sequential three-way decision.

References

[1] Y. Y. Yao, "Three-way decisions with probabilistic rough sets," *Inf. Sci.*, vol. 180, no. 3, pp. 341–353, 2010.

[2] S. Y. Yang, H. Y. Zhang, B. D. Baets, M. Jah, and G. Shi, "Quantitative dominance-based neighborhood rough sets via fuzzy preference relations," *IEEE Trans. Fuzzy Syst.*, 2019, doi: 10.1109/TFUZZ.2019.2955883.

[3] B. Huang, W. Z. Wu, J. J. Yan, H. X. Li, and X. Z. Zhou, "Inclusion measure-based multi-granulation decision-theoretic rough sets in multi-scale intuitionistic fuzzy information tables," *Inf. Sci.*, vol. 507, pp. 421–448, 2020.

[4] A. H. Tan, W. Z. Wu, Y. H. Qian, J. Y. Liang, J. K. Chen, and J. J. Li, "Intuitionistic fuzzy rough set-based granular structures and attribute subset selection," *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 3, pp. 527–539, 2019.

[5] J. T. Wang, Y. H. Qian, F. J. Li, J. Y. Liang, and W. P. Ding, "Fusing fuzzy monotonic decision trees," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 5, pp. 887–900, 2020.

[6] W. P. Ding, W. Pedrycz, I. Triguero, Z. H. Cao, and C. T. Lin, "Multigranulation super-trust model for attribute reduction," *IEEE Trans. Fuzzy Syst.*, 2020, doi: 10.1109/TFUZZ.2020.2975152.

[7] L. Sun, L. Y. Y. Wang, W. P. Ding, Y. H. Qian, and J. C. Xu, "Feature selection using fuzzy neighborhood entropy-based uncertainty measures for fuzzy neighborhood multigranulation rough sets," *IEEE Trans. Fuzzy Syst.*, 2020, doi: 10.1109/TFUZZ.2020.2989098.

[8] Y. H. Qian, X. Y. liang, G. P. Lin, Q. Guo, and J. Y. Liang, "Local multigranulation decision-theoretic rough sets," *Int. J. Approx. Reason.*, vol. 82, pp. 119–137, 2017.

[9] C. Z. Wang, Y. Wang, S. M. Wen, Y. H. Qian, and D. G. Chen, "Fuzzy rough attribute reduction for categorical data," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 5, pp. 818–830, 2020.

[10] Q. Z. Kong, X. W. Zhang, W. H. Xu, and S. T. Xie, "Attribute reducts of multi-granulation information system," *Artif. Intell. Rev.*, vol. 53, no. 2, pp. 1353–1371, 2020.

[11] L. Yang, W. H. Xu, X. Y. Zhang, and B. B. Sang, "Multi-granulation method for information fusion in multi-source decision information system," *Int. J. Approx. Reason.*, vol. 122, pp. 47–65, 2020.

[12] W. H. Xu and W. T. Li, "Granular computing approach to two-way learning based on formal concept analysis in fuzzy datasets," *IEEE Trans. Cybern.*, vol. 46, no. 2, pp. 366–379, 2016.

[13] W. H. Xu and J. H. Yu, "A novel approach to information fusion in multi-source datasets: A granular computing viewpoint," *Inf. Sci.*, vol. 378, pp. 410–423, 2017.

[14] J. T. Yao and N. Azam, "Web-based medical decision support systems for three-way medical decision making with game-theoretic rough sets," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 1, pp. 3–15, 2015.

[15] J. H. Li, C. C. Huang, J. J. Qi, Y. H. Qian, and W. Q. Liu, "Three-way cognitive concept learning via multi-granularity," *Inf. Sci.*, vol. 378, pp. 244–263, 2017.

[16] D. Liu and X. Q. Yue, "A matrix factorization based dynamic granularity recommendation with three-way decisions," *Knowl. Based Syst.*, vol. 191, 2020, 105243.

[17] H. Yu, Z. H. Chang, G. Y. Wang, and X. F. Chen, "An efficient three-way clustering algorithm based on gravitational search," *Int. J. Mach. Learn. Cyb.*, vol. 11, no. 5, pp. 1003–1016, 2020.

[18] H. X. Li, L. B. Zhang, B. Huang, and X. Z. Zhou, "Sequential three-way decision and granulation for cost-sensitive face recognition," *Knowl. Based Syst.*, vol. 91, pp. 241–251, 2016.

[19] X. Y. Jia, Z. Deng, F. Min, and D. Liu, "Three-way decisions based feature fusion for chinese irony detection," *Int. J. Approx. Reason.*, vol. 113, pp. 324–335, 2019.

[20] J. M. Zhan, H. B. Jiang, and Y. Y. Yao, "Three-way multi-attribute decision-making based on outranking relations," *IEEE Trans. Fuzzy Syst.*, 2020, doi: 10.1109/TFUZZ.2020.3007423.

[21] A. Campagner, F. Cabitza, and D. Ciucci, "Three-way decision for handling uncertainty in machine learning: A narrative review," in *Int. Joint Conf. Rough Sets*, 2020, pp. 137–152.

[22] Q. H. Zhang, Y. H. Chen, J. Yang, and G. Y. Wang, "Fuzzy entropy: A more comprehensible perspective for interval shadowed sets of fuzzy sets," *IEEE Trans. Fuzzy Syst.*, 2019, doi: 10.1109/TFUZZ.2019.2947224.

[23] H. X. Li, L. B. Zhang, B. Huang, and X. Z. Zhou, "Cost-sensitive dual-bidirectional linear discriminant analysis," *Inf. Sci.*, vol. 510, pp. 283–303, 2020.

[24] B. H. Long, W. H. Xu, X. Y. Zhang, and L. Yang, "The dynamic update method of attribute-induced three-way granular concept in formal contexts," *Int. J. Approx. Reason.*, vol. 126, pp. 228–248, 2020.

[25] Y. Y. Yao, "Three-way conflict analysis: Reformulations and extensions of the pawlak model," *Knowl. Based Syst.*, vol. 180, pp. 26–37, 2019.

[26] X. D. Yue, J. Zhou, Y. Y. Yao, and D. Q. Miao, "Shadowed neighborhoods based on fuzzy rough transformation for three-way classification," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 5, pp. 978–991, 2020.

[27] F. Jia and P. D. Liu, "A novel three-way decision model under multiple-criteria environment," *Inf. Sci.*, vol. 471, pp. 29–51, 2019.

[28] X. Yang, T. R. Li, H. Fujita, and D. Liu, "A sequential three-way approach to multi-class decision," *Int. J. Approx. Reason.*, vol. 104, pp. 108–125, 2019.

[29] H. R. Ju, W. Pedrycz, H. X. Li, W. P. Ding, X. B. Yang, and X. Z. Zhou, "Sequential three-way classifier with justifiable granularity," *Knowl. Based Syst.*, vol. 163, pp. 103–119, 2019.

[30] H. Fujita, A. Gaeta, V. Loia, and F. Orciuoli, "Hypotheses analysis and assessment in counterterrorism activities: A method based on owa and fuzzy probabilistic rough sets," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 5, pp. 831–845, 2020.

[31] H. X. Li, L. B. Zhang, X. Z. Zhou, and B. Huang, "Cost-sensitive sequential three-way decision modeling using a deep neural network," *Int. J. Approx. Reason.*, vol. 85, pp. 68–78, 2017.

[32] D. Liu, X. Yang, and T. R. Li, "Three-way decisions: beyond rough sets and granular computing," *Int. J. Mach. Learn. Cyb.*, vol. 11, no. 5, pp. 989–1002, 2020.

[33] L. B. Zhang, H. X. Li, X. Z. Zhou, and B. Huang, "Sequential three-way decision based on multi-granular autoencoder features," *Inf. Sci.*, vol. 507, pp. 630–643, 2020.

[34] D. C. Liang, M. W. Wang, and Z. S. Xu, "Heterogeneous multi-attribute nonadditivity fusion for behavioral three-way decisions in interval type-2 fuzzy environment," *Inf. Sci.*, vol. 496, pp. 242–263, 2019.

[35] W. J. Lei, W. M. Ma, and B. Z. Sun, "Multigranulation behavioral three-way group decisions under hesitant fuzzy linguistic environment," *Inf. Sci.*, vol. 537, pp. 91–115, 2020.

[36] T. X. Wang, H. X. Li, L. B. Zhang, X. Z. Zhou, and B. Huang, "A three-way decision model based on cumulative prospect theory," *Inf. Sci.*, vol. 519, pp. 74–92, 2020.

[37] T. X. Wang, H. X. Li, X. Z. Zhou, B. Huang, and H. B. Zhu, "A prospect theory-based three-way decision model," *Knowl. Based Syst.*, vol. 203, 2020, 106129.

[38] D. E. Bell, "Regret in decision making under uncertainty," *Oper. Res.*, vol. 30, pp. 961–981, 1982.

[39] G. Loomes and R. Sugden, "Regret theory: An alternative theory of rational choice under uncertainty," *Econ. J.*, vol. 92, no. 368, pp. 805–824, 1982.

[40] C. G. Chorus, "Regret theory-based route choices and traffic equilibria," *Transportmetrica*, vol. 8, no. 4, pp. 291–305, 2012.

[41] D. C. Liang, M. W. Wang, Z. S. Xu, and X. Chen, "Risk interval-valued three-way decisions model with regret theory and its application to project resource allocation," *J. Oper. Res. Soc.*, 2019, doi: 10.1080/01605682.2019.1654939.

[42] H. G. Peng, K. W. Shen, S. S. He, H. Y. Zhang, and J. Q. Wang, "Investment risk evaluation for new energy resources: An integrated decision support model based on regret theory and electre iii," *Energy Convers. Manage.*, vol. 183, pp. 332–348, 2019.

[43] Y. Wang, "Regret-based automated decision-making aids for domain search tasks using human-agent collaborative teams," *IEEE Trans. Control Syst. Technol.*, vol. 24, no. 5, pp. 1680–1695, 2016.

[44] Y. J. Lin, Q. H. Hu, J. H. Liu, J. J. Li, and X. D. Wu, "Streaming feature selection for multilabel learning based on fuzzy mutual information," *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 6, pp. 1491–1507, 2017.

[45] W. P. Ding, C. T. Lin, M. Prasad, Z. H. Cao, and J. D. Wang, "A layered-coevolution-based attribute-boosted reduction using adaptive quantum-behavior pso and its consistent segmentation for neonates brain tissue," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 3, pp. 1177–1191, 2018.

[46] W. Yao, "A categorical isomorphism between injective stratified fuzzy t00 spaces and fuzzy continuous lattices," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 1, pp. 131–139, 2016.

[47] H. C. Liao, X. M. Mi, Z. S. Xu, J. P. Xu, and F. Herrera, "Intuitionistic fuzzy analytic network process," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 5, pp. 2578–2590, 2018.

[48] Y. H. Qian, Y. B. Li, J. Y. Liang, G. P. Lin, and C. Y. Dang, "Fuzzy granular structure distance," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 6, pp. 2245–2259, 2015.

[49] J. B. Liu, H. X. Li, X. Z. Zhou, B. Huang, and T. X. Wang, "An optimization-based formulation for three-way decisions," *Inf. Sci.*, vol. 495, pp. 185–214, 2019.

[50] Q. H. Zhang, C. C. Yang, and G. Y. Wang, "A sequential three-way decision model with intuitionistic fuzzy numbers," *IEEE Trans. Syst., Man, Cybern., Syst.*, 2019, doi: 10.1109/TSMC.2019.2908518.

[51] P. D. Liu, Y. M. Wang, F. Jia, and H. Fujita, "A multiple attribute decision making three-way model for intuitionistic fuzzy numbers," *Int. J. Approx. Reason.*, vol. 119, pp. 177–203, 2020.

[52] G. M. Lang, D. Q. Miao, and H. Fujita, "Three-way group conflict analysis based on pythagorean fuzzy set theory," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 3, pp. 447–461, 2020.

[53] D. C. Liang, Z. S. Xu, and D. Liu, "A new aggregation method-based error analysis for decision-theoretic rough sets and its application in hesitant fuzzy information systems," *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 6, pp. 1685–1697, 2017.

[54] J. M. Mendel, R. I. John, and F. L. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 6, pp. 808–821, 2006.

[55] Y. C. Xiao, B. Q. Hu, and X. R. Zhao, "Three-way decisions based on type-2 fuzzy sets and interval-valued type-2 fuzzy sets," *J. Intell. Fuzzy Syst.*, vol. 31, no. 3, pp. 1385–1395, 2016.

[56] J. M. Garibaldi, M. Jaroszewski, and S. Musikasuwan, "Nonstationary fuzzy sets," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 4, pp. 1072–1086, 2008.

[57] J. M. Garibaldi and S. Guadarrama, "Constrained type-2 fuzzy sets," in *IEEE Symp. Adv. Type-2 Fuzzy Logic Syst.*, 2011, pp. 66–73.

[58] C. Wagner, S. Miller, J. M. Garibaldi, D. T. Anderson, and T. C. Havens, "From interval-valued data to general type-2 fuzzy sets," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 2, pp. 248–269, 2015.

[59] H. Dalman and M. Bayram, "Interactive fuzzy goal programming based on taylor series to solve multiobjective nonlinear programming problems with interval type-2 fuzzy numbers," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 4, pp. 2434–2449, 2017.

[60] I. Eyoh, R. John, and G. D. Maere, "Interval type-2 a-intuitionistic fuzzy logic for regression problems," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 4, pp. 2396–2408, 2018.

[61] D. C. Liang, Z. S. Xu, D. Liu, and Y. Wu, "Method for three-way decisions using ideal topsis solutions at pythagorean fuzzy information," *Inf. Sci.*, vol. 435, pp. 282–295, 2018.

[62] J. D. Qin and X. W. Liu, "Multi-attribute group decision making using combined ranking value under interval type-2 fuzzy environment," *Inf. Sci.*, vol. 297, pp. 293–315, 2015.

[63] S. M. Chen and J. A. Hong, "Fuzzy multiple attributes group decision-making based on ranking interval type-2 fuzzy sets and the topsis method," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 44, no. 12, pp. 1665–1673, 2014.

[64] S. M. Chen and L. W. Lee, "Fuzzy multiple attributes group decision-making based on the interval type-2 topsis method," *Expert Syst. Appl.*, vol. 37, no. 4, pp. 2790–2798, 2010.

[65] S. L. Liu and X. W. Liu, "A prospect theory-based method for linguistic decision making under risk," *J. Southeast Univ.*, vol. 33, no. 3, pp. 370–376, 2017.

[66] J. H. Hu, Y. Zhang, X. H. Chen, and Y. M. Liu, "Multi-criteria decision making method based on possibility degree of interval type-2 fuzzy number," *Knowl. Based Syst.*, vol. 43, pp. 21–29, 2013.

[67] J. Quiggin, "Regret theory with general choice sets," *J. Risk Uncertainty*, vol. 8, no. 2, pp. 153–165, 1994.

[68] S. M. Chen and L. W. Lee, "Fuzzy multiple attributes group decision-making based on the ranking values and the arithmetic operations of interval type-2 fuzzy sets," *Expert Syst. Appl.*, vol. 37, no. 1, pp. 824–833, 2010.

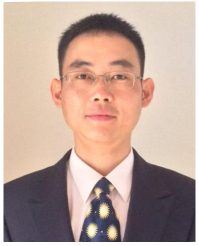
[69] R. Wang, J. H. Cheng, Y. L. Zhu, and W. W. Xiong, "Research on diversity of mineral resources carrying capacity in chinese mining cities," *Resour. Policy*, vol. 47, pp. 108–114, 2016.

[70] C. Wang, H. J. Geng, Z. L. Shui, and H. L. Yao, "China's urban minerals policies: Evolution, problems and countermeasures-a quantitative research," *J. Clean. Prod.*, vol. 197, no. 1, pp. 114–123, 2018.

[71] D. C. Liang, W. Pedrycz, and D. Liu, "Determining three-way decisions with decision-theoretic rough sets using a relative value approach," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 8, pp. 1785–1799, 2017.

Tianxing Wang received the B.E. degree from Sichuan University, Chengdu, China, in 2014, and the M.E. degree from Nanjing University, Nanjing, China, in 2017. He is currently pursuing the Ph.D. degree in management science and engineering with Nanjing University, Nanjing, China. His main research interests include three-way decision theory, fuzzy set theory, granular computing, and intelligent information processing.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2020.3033448, IEEE Transactions on Fuzzy Systems



Huaxiong Li (M'11) received the M.E. degree in control theory and control engineering from Southeast University, Nanjing, China, in 2006, and Ph.D. degree from Nanjing University, Nanjing, China, in 2009. He is currently an Associate Professor with the Department of Control and Systems Engineering, Nanjing University, Nanjing, China. He was a visiting scholar at the Department of Computer Science, University of Regina, Canada, from 2007 to 2008, and a visiting scholar at the University of Hong Kong, Hong Kong, China, in 2010. His current research interests include granular computing, three-way decision, knowledge discovery, and machine learning. He is a Committee Member of China Association of Artificial Intelligence (CAAI) Granular Computing and Knowledge Discovery (GCKD) Committee, and a Committee Member of China Association of Artificial Intelligence (CAAI) Machine Learning Committee.

Yuhua Qian received the M.S. and Ph.D. degrees in computers with applications from Shanxi University, Taiyuan, China, in 2005 and 2011, respectively. He is currently a Professor with the Institute of Big Data Science and Industry, and also with the Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Shanxi University. He is best known for multigranulation rough sets in learning from categorical data and granular computing. He is involved in research on pattern recognition, feature selection, rough set theory, granular computing, and artificial intelligence. He has authored over 80 articles on these topics in international journals. He has served on the Editorial Board of the *International Journal of Knowledge-Based Organizations* and *Artificial Intelligence Research*. He has served as the Program Chair or Special Issue Chair of the Conference on Rough Sets and Knowledge Technology, the Joint Rough Set Symposium, and the Conference on Industrial Instrumentation and Control, and a PC Member of many machine learning, data mining, and granular computing conferences.

Bing Huang received the M.S. degree from Sichuan University, Chengdu, China, in 1999, and the Ph.D. degree from Nanjing University of Scince and Technology, Nanjing, China, in 2005. He is currently a Professor with the School of Information Engineering, Nanjing Audit University, Nanjing, China. His current research interests include rough sets and granular computing.

Xianzhong Zhou (M'10) received the B.S. and M.S. degrees in system engineering and the Ph.D. degree in control theory and application from the Nanjing University of Science and Technology, Nanjing, China, in 1982, 1985, and 1996, respectively. He is currently a Professor of Control Science and Engineering, School of Management and Engineering, Nanjing University, Nanjing. His research interests include intelligent information processing and intelligent systems (novel decision-making system theory based on humanware technology, rough set theory, and applications), modeling, control, and optimization of complex systems, and future integrated automation systems. Prof. Zhou was among the people selected for "333 Engineering" of Jiangsu Province in 2002 and 2007 and Excellent Science and Technology Worker of Jiangsu Province in 2000. He is the Executive Director of Systems Engineering Society of China and the Vice President of Systems Engineering Society of Jiangsu Province.