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A multiple-valued logic approach for multigranulation rough set model ☆,☆☆

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A B S T R A C T

Rough sets have often been studied under a three-valued logic framework. In this paper, we attempt to extend the previous study in two ways. Firstly, we extend the previous study from single-granulation to multigranulation. Secondly, we study multigranulation rough set theory from the viewpoint of three-way decision. More precisely, we embody the idea of three-way decision theory in the definition of multigranulation rough set theory. This leads to an axiomatic definition of decision-oriented aggregation operators on $\mathbf{3} = \{0, \frac{1}{2}, 1\}$, which are quite different from those conjunctions proposed so far. Moreover, considering that a multigranulation rough set also divides the universe into five disjoint subsets, we present a five-valued semantics for multigranulation rough set model, and a kind of non-deterministic matrices is thus given.

1. Introduction

The notion of multigranulation rough set model [20,28] is derived from a multigranulation space $(U, \mathbf{E})$, which consists of a universe U and a family of equivalence relations $\mathbf{E}$ on U. To date, rough set models in multigranulation spaces have become a subject of growing interests in artificial intelligence and the related areas. According to the combination strategies used in the existing studies, the existing rough set models in multigranulation spaces can be classified into the following two types [28]. One model is based on a combination of a family of equivalence relations into an equivalence relation and the construction of approximations with respect to the combined relation. For instance, Marek and Rasiowa and Polkowski considered gradual approximations of sets by using a nested sequence of equivalence relations [15,17]. Yao studied hierarchical multigranulation space and the induced stratified rough set approximations [29,30]. Fariñas Del Cerro and Orłowska used both intersection and transitive closure of union of equivalence relations in a logic for data analysis [7]. Rauszer presented a formal system for reasoning with incomplete information in a multi-agent system [22]. The other approach has been taken up along the research line of the proposed multigranulation rough set models in [20], that is, the rough set models are obtained by combining the family of rough sets produced by the individual approximation spaces in a multigranulation space. The derived models include fuzzy multigranulation rough set model, decision-theoretic rough sets, see, e.g., [10,14,21,25,26,32]. At the application level, Liang et al. proposed an efficient feature selection algorithm for

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large-scale data sets based on ideas from multigranulation rough sets [13]. Li focused on three-way concept learning via multi-granularity from the viewpoint of cognition [12]. Yang et al. proposed a test cost sensitive multigranulation rough set model [27]. For the most recently published work on this topic, one can refer to [23] for detail.

Rough sets have often been studied under a three-valued logic framework and different authors have tried to connect rough sets to different logics [1–6,8,16,18]. Three-valued logics are straightforward generalizations of Boolean logic based on the most simple bipolar scale $\{0, \frac{1}{2}, 1\}$ where 1 (resp. 0) has a positive (resp. negative) flavor, and $\frac{1}{2}$ is neutral. There have been several different meanings attached to the third value, which correspond to different types of three-valued logical systems. However, these three-valued logical systems still form a scattered landscape, Ciucci and Dubois defined a maximal family of sensible conjunctions [4] on **3** based on some intuitive properties, in the scope of modeling incomplete information. Consequently, 14 possible cases of conjunctions are obtained.

In this paper, we aim to investigate multigranulation rough set theory from the viewpoint of multiple-valued logics. Considering the fact that a multigranulation rough set can induce a three-valued function on the universe, we define various types of multigranulation rough set models according to the proposed 14 conjunctions on $\mathbf{3} = \{0, \frac{1}{2}, 1\}$. Consequently, 14 types of multigranulation rough set models are obtained. Then, a comparative study between the models in the existing literatures and the proposed ones in the present paper is performed. The results show that some existing ones can be put into our framework from the viewpoint of 3-valued logic.

Note that a common feature shared by the conjunctions in [4] that $1 * 0 = 0 * 1 = 0$, i.e., they are generalizations of Boolean conjunction. However, if the values in $\{0, \frac{1}{2}, 1\}$ are interpreted in terms of three-way decision [29,31,32], the proposed conjunctions have some drawbacks. More concretely, if we interpret 1 as the acceptance value while 0 as the rejection value, then the pair $(1, 0)$ means that an object should be accepted according to one evaluation function and be rejected according to the other evaluation function simultaneously. In such a situation, one always makes noncommitment according to the basic idea of three-way decision. To embody the idea of three-way decision in the definition of multigranulation rough set models, we firstly present a modified version of conjunctions on **3**. Then by transferring the decision-oriented aggregation operators on three-valued functions into combination strategy of approximation results in different Pawlak spaces, we define 16 types of multigranulation rough set models. Lastly, considering that any multigranulation rough set model can also divide the universe into five disjoint subsets, we present a five-valued semantic for multigranulation rough set model and show its nondeterministic character.

The rest of this paper proceeds as follows. In Section 2, by recalling the existing conjunctions on **3**, we define the corresponding 14 types of multigranulation rough set models. In Section 3, to embody the idea of three-way decision, we present an axiomatic definition of decision-oriented aggregation operators on **3** and derive 16 multigranulation rough set models in a similar manner as that in Section 2. Then in Section 4, we present a five-valued logic semantics for multigranulation rough set, such a semantic is defined by using a non-deterministic logical matrix [2] (Nmatrix), which is a generalization of an ordinary matrix modeling non-determinism, with interpretations of logical connectives returning sets of logical values instead of single values. Lastly, the present paper is completed with some concluding remarks.

2. Multigranulation rough set and three-valued logic

In this section, we aim to develop some new multigranulation rough set models by means of the existing conjunction operators on $\mathbf{3} = \{0, \frac{1}{2}, 1\}$. We restrict our discussion to the models derived from a multigranulation space $(U, \mathbf{E})$, where only two equivalence relations are available, that is, $|\mathbf{E}| = 2$. The present study can be extended to a large number of equivalence relations by using the associative law.

2.1. Approximation in a Pawlak approximation space

A Pawlak space is a pair (U, R), where R is an equivalence relation on U. The equivalence relation induces a partition U/R, namely, a family of pairwise disjoint nonempty subsets of the universe whose union is the universe. The equivalence class containing x is given by $[x]_R = \{y \in U | xRy\}$. By considering equivalence classes as the building blocks, one can approximate a subset of U through unions of equivalence classes.

Definition 1 *([19])*. In a Pawlak approximation space (U, R), a subset $X \subseteq U$ is approximated by two sets:

$$\underline{apr}_R(X) = \cup\{[x]_R \mid [x]_R \subseteq X\},$$
$$\overline{apr}_R(X) = \cup\{[x]_R \mid [x]_R \cap X \neq \emptyset\}.$$

Equivalently, they can be defined by:

$$\underline{apr}_R(X) = \{x \in U \mid [x]_R \subseteq X\},$$
$$\overline{apr}_R(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}.$$

We call $\underline{apr}_R(X), \overline{apr}_R(X)$ the lower approximation and upper approximation of X, respectively, with respect to the equivalence relation R and call $\underline{apr}_R, \overline{apr}_R : 2^U \to 2^U$ rough approximation operators determined by R.

Pawlak rough sets have often been studied under a three-valued logic framework and different authors have tried to connect rough sets to different logics. Consider the most simple bipolar scale $\{0, \frac{1}{2}, 1\}$, where 1 (resp. 0) has a positive (resp. negative) flavor, and $\frac{1}{2}$ is neutral. Let f be a three-valued function on the universe U, then f can induce three (Boolean) subsets of the universe:

$A_1 = \{x \mid f(x) = 1\}$, the positive domain,
$A_0 = \{x \mid f(x) = 0\}$, the negative domain,
$A_u = \{x \mid f(x) = \frac{1}{2}\}$, the neutral domain.

Clearly, (A_1, A_0, A_u) form a tripartition of the universe. Conversely, given a tripartition of the universe, say as (A_1, A_0, A_u), we can define a three-valued function in an obvious way: $f(x) = 1$ if $x \in A_1$, $f(x) = 0$ if $x \in A_0$ and $f(x) = \frac{1}{2}$ if $x \in A_u$. Thus the collection of three-valued functions on the universe and the collection of tripartitions are in one-to-one correspondence.

We now turn our attention to Pawlak space (U, R) and $X \subseteq U$, rough set approximation of X naturally induces a tripartition of the universe, i.e., $(\underline{apr}_R(X), \overline{apr}_R(X) \setminus \underline{apr}_R(X), \overline{apr}^c_R(X))$, which, in turn, induces a three-valued function, denoted by $f_{X,R}$, on the universe, which is defined as follows:

$f_{X,R}(x) = 1$, if $x \in \underline{apr}_R(X)$,
$f_{X,R}(x) = 0$, if $x \in (\overline{apr}_R(X))^c$,
$f_{X,R}(x) = \frac{1}{2}$, if $x \in \overline{apr}_R(X) \setminus \underline{apr}_R(X)$.

2.2. The existing multigranulation rough set models

Recall that in [20], Qian et al. proposed the notion of multigranulation rough set models from the viewpoint of granular computing, the results are a pair of so-called pessimistic and optimistic rough sets. By interpreting a multigranulation space as a multiple-source approximation space, Khan and Banerjee introduced the notions of strong and weak lower and upper approximations [1]. Although the interpretations of a multigranulation space in these two studies are different, their results are mathematically equivalent.

Definition 2 *([20])*. Let $(U, \mathbf{E})$ be a multigranulation space with $\mathbf{E} = \{R_1, R_2\}$, then for any $X \subseteq U$, define

$$\underline{apr}^o_{\mathbf{E}}(X) = \underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X),$$
$$\overline{apr}^o_{\mathbf{E}}(X) = \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X),$$
$$\underline{apr}^p_{\mathbf{E}}(X) = \underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X),$$
$$\overline{apr}^p_{\mathbf{E}}(X) = \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X).$$

Then we call the pair $(\underline{apr}^o_{\mathbf{E}}(X), \overline{apr}^o_{\mathbf{E}}(X))$ an optimistic multigranulation rough set and call $(\underline{apr}^p_{\mathbf{E}}(X), \overline{apr}^p_{\mathbf{E}}(X))$ a pessimistic multigranulation rough set.

Given $(U, \mathbf{E})$ and $X \subseteq U$, a multigranulation rough set (either optimistic or pessimistic) divides the universe into three disjoint subsets. i.e., the positive region, the negative region and the boundary region. The corresponding three-valued function is denoted by $f_{X,\mathbf{E}}$, which can be similarly defined as above. Consequently, for $(U, \mathbf{E})$ with $\mathbf{E} = \{R_1, R_2\}$, we obtain three types of three-valued functions on the universe for $X \subseteq U$: $f_{X,\mathbf{E}}$, f_{X,R_1} and f_{X,R_2}. Then one natural question arises: what is the internal relationship between $f_{X,\mathbf{E}}$ and f_{X,R_1}, f_{X,R_2}? Or in other words, how can we obtain $f_{X,\mathbf{E}}$ by means of operations on f_{X,R_1}, f_{X,R_2}. In particular, if we consider the pointwise operation on three-valued functions, what is the aggregation operator corresponding to the relationship between $f_{X,\mathbf{E}}$ and f_{X,R_1},,f_{X,R_2}. In what follows, we attempt to examine this issue in detail.

Proposition 1. *Let $(U, \mathbf{E})$ be a multigranulation space with $\mathbf{E} = \{R_1, R_2\}$. Then the combination operator $\otimes^o$ induced by the optimistic multigranulation rough set model satisfies the following conditions:*
(1) $0 \otimes^o 0 = 1 \otimes^o \frac{1}{2} = \frac{1}{2} \otimes^o 0 = 0$, $\frac{1}{2} \otimes^o \frac{1}{2} = \frac{1}{2}$, $\frac{1}{2} \otimes^o 1 = 1 \otimes^o \frac{1}{2} = 1 \otimes^o 1 = 1$.
(2) $\otimes^o$ *is undefined on* $(0, 1)$ *and* $(1, 0)$.

Proof. Let $f_{X,R_1}, f_{X,R_2}, f^o_{X,\mathbf{E}}$ be the three-valued functions induced by X in Pawlak spaces (U, R_1), (U, R_2) and the multigranulation space $(U, \mathbf{E})$, respectively. Considering one-to-one correspondence between the collection of three-valued functions and tripartitions on a universe, we need only show that the following conditions are satisfied:
(i) $f^o_{X,\mathbf{E}}(x) = 1$ if and only if $f_{X,R_1}(x) = 1$ or $f_{X,R_2}(x) = 1$,
(ii) $f^o_{X,\mathbf{E}}(x) = 0$ if and only if $f_{X,R_1}(x) = 0$ or $f_{X,R_2}(x) = 0$.

Table 1
Aggregation operator $\otimes^o$ induced by optimistic multigranulation rough set model.

$\otimes^o$	0	$\frac{1}{2}$	1
0	0	0	$\times$
$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	$\times$	1	1

Table 2
Aggregation operator $\otimes^p$ corresponding to pessimistic multigranulation rough set model.

$\otimes^p$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	$\times$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	$\times$	$\frac{1}{2}$	1

Since the operation on three-valued sets, we are considering now, is pointwise defined, then condition (i) is equivalent to the fact that $x \otimes^o y = 1$ if and only if at least one of $\{x, y\}$ is equal to 1, which, however, can be seen directly from Table 1. It deserves special noting here that the combination value of 1 and 0 is not defined, that is, $\otimes^o$ in Table 1 is just a partial operation, which is due to the following fact: $f_{X,R_1}(x) = 1$ and $f_{X,R_2}(x) = 0$ cannot hold simultaneously. (In fact, suppose that $f_{X,R_1}(x) = 1$, i.e., x is contained in the lower approximation of X, we then have $x \in X$, and therefore, $[x]_{R_2} \cap X \neq \emptyset$, which means that $f_{X,R_2}(x) \in \{\frac{1}{2}, 1\}$, that is, $f_{X,R_2}(x) = 0$ cannot happen.) Similarly, $f_{X,R_1}(x) = 0$ and $f_{X,R_2}(x) = 1$ cannot hold simultaneously. By using a similar way, we can show that condition (ii) is also satisfied. □

The aggregation operator $\otimes^o$ is also shown in Table 1.

Proposition 2. *Suppose that $(U, \mathbf{E})$ is a multigranulation space with $\mathbf{E} = \{R_1, R_2\}$. Then the aggregation operator induced by pessimistic multigranulation rough set model satisfies the following conditions:*
(1) $0 \otimes^p 0 = 0, 0 \otimes^p \frac{1}{2} = \frac{1}{2} \otimes^p 0 = \frac{1}{2} \otimes^p \frac{1}{2} \frac{1}{2} \otimes^p 1 = 1 \otimes^p \frac{1}{2} = \frac{1}{2}, 1 \otimes^p 1 = 1,$
(2) $\otimes^p$ *is undefined on* $(0, 1)$ *and* $(1, 0)$.

Proof. It can be shown in a similar manner as that of Proposition 1. □

The aggregation operator $\otimes^p$ is shown in Table 2.

2.3. Conjunctions on three-valued logic

In [4], the authors defined a maximal family of sensible conjunctions, in addition to the existing ones.

Definition 3 *([4]).* A conjunction on $\mathbf{3}$ is a binary mapping $* : \mathbf{3} \times \mathbf{3} \longrightarrow \mathbf{3}$, that is monotonically increasing in the wide sense, and extends the connective AND in Boolean logic:
(C1) If $x \le y$ then $x * z \le y * z$;
(C2) If $x \le y$ then $z * x \le z * y$;
(C3) $0 * 0 = 0 * 1 = 1 * 0 = 0$ and $1 * 1 = 1$.

It is pointed in [4] that there are 14 possible conjunctions satisfying conditions in Definition 3. Among them, only six are commutative and only five are associative. These five conjunctions are already known in the literature and precisely, they have been studied in the following logics: Sette, Sobocinski, Łukasiewicz, Kleene, Bochvar. The complete list is given in Table 3. The smaller table gives the common parts of all operators, and the larger table gives the rest.

We conclude from Table 1 and Table 2 that the existing multigranulation rough set models correspond to two different aggregation operators on $\mathbf{3} = \{0, \frac{1}{2}, 1\}$, which can be regarded as two special cases of conjunctions in the sense of Definition 2. Then one question arises at this point, i.e., what is the multigranulation rough set model corresponding to conjunctions in Definition 2, this leads us to a consideration of generalization of the exiting multigranulation rough set models from the viewpoint of three-valued logic.

In what follows, we use $*_i$ to denote the ith conjunction operator in Table 3, and use $(\underline{apr}^i_{\mathbf{E}}, \overline{apr}^i_{\mathbf{E}})$ to denote the corresponding multigranulation approximation operators. The obtained results are summarized in the following proposition.

Table 3
All conjunction on $\{0, \frac{1}{2}, 1\}$.

$*$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0		
1	0		1

n	$\frac{1}{2}*\frac{1}{2}$	$\frac{1}{2}*1$	$1*\frac{1}{2}$	
1	1	1	1	Sette
2	$\frac{1}{2}$	1	1	quasi-conjunction/Sobociński
3	$\frac{1}{2}$	1	$\frac{1}{2}$	
4	$\frac{1}{2}$	$\frac{1}{2}$	1	
5	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	min/interval conjunction/Kleene
6	0	0	1	
7	0	0	$\frac{1}{2}$	
8	0	0	0	Bochvar external
9	0	$\frac{1}{2}$	0	
10	0	$\frac{1}{2}$	1	
11	0	$\frac{1}{2}$	$\frac{1}{2}$	Łukasiewicz
12	0	1	0	
13	0	1	$\frac{1}{2}$	
14	0	1	1	

Proposition 3. *Suppose that $(U, \{R_1, R_2\})$ is a multigranulation space, then for $X \subseteq U$, we have*

(1) $(\underline{apr}^1_{\mathbf{E}}(X), \overline{apr}^1_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X))$.
(2) $(\underline{apr}^2_{\mathbf{E}}(X), \overline{apr}^2_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X))$.
(3) $(\underline{apr}^3_{\mathbf{E}}(X), \overline{apr}^3_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X))$.
(4) $(\underline{apr}^4_{\mathbf{E}}(X), \overline{apr}^4_{\mathbf{E}}(X)) = (\underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X))$.
(5) $(\underline{apr}^5_{\mathbf{E}}(X), \overline{apr}^5_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X))$.
(6) $(\underline{apr}^6_{\mathbf{E}}(X), \overline{apr}^6_{\mathbf{E}}(X)) = (\underline{apr}_{R_2}(X), \underline{apr}_{R_2}(X))$.
(7) $(\underline{apr}^7_{\mathbf{E}}(X), \overline{apr}^7_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \underline{apr}_{R_2}(X))$.
(8) $(\underline{apr}^8_{\mathbf{E}}(X), \overline{apr}^8_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X))$.
(9) $(\underline{apr}^9_{\mathbf{E}}(X), \overline{apr}^9_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \underline{apr}_{R_1}(X))$.
(10) $(\underline{apr}^{10}_{\mathbf{E}}(X), \overline{apr}^{10}_{\mathbf{E}}(X)) = (\underline{apr}_{R_2}(X), \underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X))$.
(11) $(\underline{apr}^{11}_{\mathbf{E}}(X), \overline{apr}^{11}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X))$.
(12) $(\underline{apr}^{12}_{\mathbf{E}}(X), \overline{apr}^{12}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X), \underline{apr}_{R_1}(X))$.
(13) $(\underline{apr}^{13}_{\mathbf{E}}(X), \overline{apr}^{13}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X), \underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X))$.
(14) $(\underline{apr}^{14}_{\mathbf{E}}(X), \overline{apr}^{14}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X), \underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X))$.

Proof. We will give the proof of (2) and (6) below, and the others can be proved in an analogous manner.

(2) Bearing in mind that there exists a one-to-one correspondence between the collection of three-valued functions and that of tripartitions on the universe, we have

$$f^{-1}_{X,\mathbf{E}}(1) = \underline{apr}^2_{\mathbf{E}}(X),\ f^{-1}_{X,\mathbf{E}}(0) = (\overline{apr}^2_{\mathbf{E}}(X))^c, \tag{1}$$

$$f^{-1}_{X,R_1}(1) = \underline{apr}_{R_1}(X),\ f^{-1}_{X,R_1}(0) = (\overline{apr}_{R_1}(X))^c, \tag{2}$$

$$f^{-1}_{X,R_2}(1) = \underline{apr}_{R_2}(X),\ f^{-1}_{X,R_2}(0) = (\overline{apr}_{R_2}(X))^c. \tag{3}$$

According to Table 3, for $x, y \in \{0, \frac{1}{2}, 1\}$, $x *_2 y = 1$ if and only if at least one of $\{x, y\}$ is equal to 1, $x *_2 y = 0$ if and only if at least one of $\{x, y\}$ is equal to 0. Considering that the operation between f_{X,R_1} and f_{X,R_2} is pointwise defined, we have

$$f_{X,\mathbf{E}}(x) = 1 \Leftrightarrow f_{R_1,X}(x) = 1 \text{ or } f_{R_2,X}(x) = 1, \tag{4}$$

And therefore,

$$x \in f^{-1}_{X,\mathbf{E}}(1) \Leftrightarrow x \in f^{-1}_{X,R_1}(1) \cup f^{-1}_{X,\mathbf{E}}(1). \tag{5}$$

Combining with (1)–(3), we obtain

$$\underline{apr}^2_{\mathbf{E}}(X) = \underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X).$$

Similarly, according to Table 3 and the pointwise operation between f_{X,R_1} and f_{X,R_2}, we have

$$f_{X,\mathbf{E}}(x) = 0 \Leftrightarrow f_{X,R_1}(x) = 0 \text{ or } f_{X,R_2}(x) = 0. \tag{6}$$

Therefore,

$$x \in f^{-1}_{X,\mathbf{E}}(0) \Leftrightarrow x \in f^{-1}_{R_1,X}(0) \cup f^{-1}_{X,\mathbf{E}}(0) = (\overline{apr}_{R_1}(X))^c \cup (\overline{apr}_{R_2}(X))^c = (\overline{apr}_{R_1}(X) \cap (\overline{apr}_{R_2}(X))^c. \tag{7}$$

Combining with (1)–(3), we therefore obtain

$$\overline{apr}^2_{\mathbf{E}}(X) = \overline{apr}_{R_1}(X)) \cap (\overline{apr}_{R_2}(X),$$

as desired.

(6) Observe from Table 3 that for $x, y \in \{0, \frac{1}{2}, 1\}$, $x *_6 y = 1$ if and only if $x = 1, y \neq 0$, $x *_6 y = 0$ if and only if $x *_6 y \neq 1$. Considering that the operation between f_{X,R_1} and f_{X,R_2} is pointwise defined, we have

$$f_{X,\mathbf{E}}(x) = 1 \Leftrightarrow f_{R_1,X}(x) = 1 \text{ and } f_{R_2,X}(x) \neq 0. \tag{8}$$

And therefore,

$$x \in f^{-1}_{X,\mathbf{E}}(1) \Leftrightarrow x \in f^{-1}_{X,R_1}(1) \cap (f^{-1}_{X,R_2}(0))^c. \tag{9}$$

Combining with (1)–(3), we obtain

$$\underline{apr}^6_{\mathbf{E}}(X) = \underline{apr}_{R_1}(X) \cap (\overline{apr}_{R_2}(X))^c. \tag{10}$$

Since $\underline{apr}_{R_1}(X) \subseteq X \subseteq \overline{apr}_{R_2}(X)$ always holds, (10) can also be written as

$$\underline{apr}^6_{\mathbf{E}}(X) = \underline{apr}_{R_1}(X). \tag{11}$$

Moreover, considering that $x *_6 y = 0$ if and only if $x *_6 y \neq 1$, we obtain $f_{X,\mathbf{E}}(x) = 0 \Leftrightarrow f_{X,\mathbf{E}}(x) \neq 1$, consequently,

$$(\overline{apr}^6_{\mathbf{E}}(X))^c = (\underline{apr}_{\mathbf{E}}(X))^c, \tag{12}$$

which, together with (11), implies that

$$\overline{apr}^6_{\mathbf{E}}(X) = \underline{apr}_{R_1}(X). \quad \square \tag{13}$$

We conclude from Proposition 3 that the multigranulation rough set model derived from the conjunction operator $*_2$ is indeed the optimistic multigranulation rough set model in [20]. However, the pessimistic multigranulation rough set model is not in the list of Proposition 3, because the corresponding aggregation operator is not the generalization of Boolean conjunction, as we will see in the next section.

We can also make the following observations: For the pair of approximation operators $(\underline{apr}^1_{\mathbf{E}}, \overline{apr}^1_{\mathbf{E}})$ and $X \subseteq U$, the boundary region of X is always empty due to the fact that the lower approximation and the upper approximation of X coincide with each other. The multigranulation rough set models corresponding to $*_3$ and $*_4$ are one-sided in the sense that the lower approximation of X is not the combination of $\underline{apr}_{R_1}(X)$ and $\underline{apr}_{R_2}(X)$, but only one of them. For the multigranulation rough set model corresponding to $*_5$, both the lower approximation and the upper approximation are obtained by using the set-theoretic intersection. They are not dual to each other in the general case. The multigranulation rough set model corresponding to $*_6$ is somewhat surprising since it has nothing to do with the approximation result in Pawlak space (U, R_1). Moreover, the boundary region for any $X \subseteq U$ is always empty. The multigranulation rough set models corresponding to $*_7$ and $*_9$ are also partially one-sided because the upper approximation has nothing to do with $\overline{apr}_{R_1}(X)$ and $\overline{apr}_{R_2}(X)$. The multigranulation rough set model derived from $*_8$ is obtained by using the set-theoretic intersection of lower approximations of X in each Pawlak space. Moreover, the approximation pair of X in multigranulation rough set model corresponding to $*_{10} - *_{14}$ is also irrelevant to the upper approximations in each Pawlak space.

3. Three-way decision and three-valued logic

The conjunctions on $\{0, \frac{1}{2}, 1\}$ corresponding to the multigranulation rough set models in last section are all partial ones in the sense that $x * y$ is definable if and only if $(x, y) \notin \{(0, 1), (1, 0)\}$. This is attributed to the fact that for any object in the universe, it cannot belong to the positive region of X in one Pawlak space and lie in the negative region in the other space simultaneously.

However, in three-way decision theory [30] (see the following definition), such a phenomenon commonly exists. For the same object, one possible situation is that its evaluation value for acceptance is 1 while the evaluation value for rejection is 0. In multi-agent environment, one agent makes acceptance while the other agent makes rejection, etc. Do the existing conjunctions suffice for this purpose? We will discuss this topic in more detail in this section.

Table 4
All decision-oriented conjunctions on $\{0, \frac{1}{2}, 1\}$.

$*$	0	$\frac{1}{2}$	1
0	0		$\frac{1}{2}$
$\frac{1}{2}$		$\frac{1}{2}$	
1	$\frac{1}{2}$		1

n	$0*\frac{1}{2}$	$\frac{1}{2}*0$	$\frac{1}{2}*1$	$1*\frac{1}{2}$
1	0	0	1	1
2	0	0	$\frac{1}{2}$	$\frac{1}{2}$
3	0	0	$\frac{1}{2}$	1
4	0	0	1	$\frac{1}{2}$
5	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
6	0	$\frac{1}{2}$	$\frac{1}{2}$	1
7	0	$\frac{1}{2}$	1	$\frac{1}{2}$
8	0	$\frac{1}{2}$	1	1
9	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
10	$\frac{1}{2}$	0	$\frac{1}{2}$	1
11	$\frac{1}{2}$	0	1	$\frac{1}{2}$
12	$\frac{1}{2}$	0	1	1
13	$\frac{1}{2}$	$\frac{1}{2}$	1	1
14	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
15	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
16	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

3.1. Three-way decision theory

In [31], based on the notions of acceptance, rejection and noncommitment, Yao proposed a theory of three-way decisions, which is an extension of the commonly used binary-decision model with an added third option.

The following definition concerns the three-way decision model derived by two evaluation functions.

Definition 4 *([30])*. Let $\emptyset \neq L_a^+ \subseteq L_a$ be a subset of L_a called the designated values for acceptance, and $\emptyset \neq L_r^- \subseteq L_r$ be a subset of L_r called the designated values for rejection. The positive, negative, and boundary regions of three-way decisions induced by (v_a, v_r) are defined by:

$$POS_{(v_a,v_r)}(v_a, v_r) = \{x \in U \mid v_a(x) \in L_a^+, v_r(x) \notin L_r^-\},$$

$$NEG_{(v_a,v_r)}(v_a, v_r) = \{x \in U \mid v_a(x) \notin L_a^+, v_r(x) \in L_r^-\},$$

$$BND_{(v_a,v_r)}(v_a, v_r) = \{x \in U \mid v_a(x) \in L_a^+, v_r(x) \in L_r^-\} \cup \{x \in U \mid v_a(x) \notin L_a^+, v_r(x) \notin L_r^-\}.$$

We now turn our attention to the complete list of conjunctions on $\{0, \frac{1}{2}, 1\}$ in Table 3. A common feature of these conjunctions is that $0*1 = 1*0 = 0*0 = 0$ and $1*1 = 1$, which is naturally the generalization of Boolean conjunction. However, if these values are interpreted in terms of three-way decision theory, such conjunctions have limitations. Precisely, if 1 indicates the acceptance of an object for one agent while 0 stands for the rejection of the same object for the other agent, then $1*0 = 0$ means that the combined decision is rejection. This does not comply with the basic idea of three-way decision theory because one always makes noncommitment in such a situation. To embody the idea of three-way decision in the definition of multigranulation rough set model, we further give an axiomatic definition of decision-oriented aggregation operators on $\mathbf{3} = \{0; \frac{1}{2}; 1\}$. As will be shown below, these operators, compared with the existing ones, share a common feature that $1*0 = 0*1 = \frac{1}{2}*\frac{1}{2} = \frac{1}{2}$. This leads us to give the following definition of decision-oriented aggregation operators.

Definition 5. A decision-oriented aggregation operator on $\mathbf{3} = \{0, \frac{1}{2}, 1\}$ is a binary mapping $* : \mathbf{3} \times \mathbf{3} \to \mathbf{3}$ satisfying the following conditions:
(i) If $x \leq y$, then $x*z \leq y*z$,
(ii) If $x \leq y$, then $z*x \leq z*y$,
(iii) $0*0 = 0, 1*1 = 1, 0*1 = 1*0 = \frac{1}{2}*\frac{1}{2} = \frac{1}{2}$.

In condition (iii), $0*1 = 1*0 = \frac{1}{2}$ means that if an object lies in the acceptance region for one evaluation function and the rejection region for the other evaluation function simultaneously, then we will make noncommitment. Similarly, $\frac{1}{2}*\frac{1}{2} = \frac{1}{2}$ means that if an object lies in the boundary region for two evaluation functions, then we will also make noncommitment. Both comply with the original idea of three-way decision in Definition 4.

A complete list of all decision-oriented aggregation operators on $\mathbf{3}$ is given in Table 4.

In what follows, some intuitional interpretations as well as several examples of the above aggregation operators are provided.

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For the first decision-oriented aggregation operator, we can give a more physical interpretation. Consider two logically sophisticated agents, for the same object in the universe, $0 * \frac{1}{2} = 0$ means that if one agent rejects it while the other agent makes noncommitment, then the combined decision is rejection. $1 * \frac{1}{2} = 1$ means that if one agent accepts an object while the other agent makes noncommitment, then the combined decision is acceptance. Similar interpretations can be given to the other cases. We can also say that the first three-way decision model is optimistic in both making acceptance and rejection, because for two agents, if one makes acceptance while the other does not make rejection, then the combined decision is acceptance. Similar decisions also hold for rejection. For the second three-way decision model, it is optimistic in making rejection while it is pessimistic in making acceptance. That is, the combined decision is acceptance if and only if both agents accept an object. For the third three-way decision model, it is optimistic in making rejection. When making acceptance, we adopt a one-sided approach, that is, when the first (the first variable represent the decision value of the first agent) agent makes acceptance while the other agent does not make rejection, then the combined decision is acceptance. In other words, the decision making of acceptance is mainly determined by the first agent. We omit here the detailed interpretation of the other decision models.

Example 1. Consider the issue of regular paper review process. Suppose that there are two reviewers, then we will see below that some types of decision results of the editorial board can be described by the above aggregation operators. $0 * 0 = 0$ means that if both reviewers recommend rejection of the paper, then the final decision of this paper is also rejection. Similarly, $1 * 1 = 1$ means that if both reviewers recommend accept of the paper, then the final decision of this paper is also accept. $0 * 1 = 1 * 0 = \frac{1}{2}$ means that if one reviewer recommends rejection while the other reviewer recommends accept, then the paper is transferred to another round of reviewer process, and therefore, $\frac{1}{2}$ is a more reasonable aggregated value of 1 and 0. $\frac{1}{2} * \frac{1}{2} = \frac{1}{2}$ means that if both reviewers do not make any recommendation, then the editorial board has to transfer it to another round of reviewer process. The above argument provides an intuitive interpretation of the axiomatic definition of decision-oriented decision operator. Then based on this, some types of decision-oriented operators in Table 4 can be interpreted as follows:

(1) The first operator: $0 * \frac{1}{2} = \frac{1}{2} * 0 = 0$, $1 * \frac{1}{2} = \frac{1}{2} * 1 = 1$ means that if one reviewer recommends rejection while the other reviewer does not recommend accept (i.e., reject or makes non-recommendation), then the final decision is rejection. Similarly, if one reviewer recommends accept while the other reviewer does not recommend rejection (i.e., accept or makes non-recommendation), then the final decision is accept.

(ii) The second decision-oriented operator: $0 * \frac{1}{2} = \frac{1}{2} * 0 = 0$ can be interpreted in a similar way as above. For $1 * \frac{1}{2} = \frac{1}{2} * 1 = \frac{1}{2}$, we combine with $\frac{1}{2} * \frac{1}{2} = \frac{1}{2}$ that if one reviewer does not recommend accept, then the paper have to be transferred to another round of review process. Or in other words, only if both reviewers recommend to accept, can the final decision be accepted. This is a strict or pessimistic attitude.

(iii) For the third aggregation operator, it is not commutative. The aggregate value is equal to 0 if and only if at least one of the aggregated value is equal to 0. That is, the final decision of a manuscript is rejection if and only if one of the reviewers recommends rejection. $\frac{1}{2} * 1 = \frac{1}{2}$, $1 * \frac{1}{2} = 1$ means that the decision to accept a manuscript mainly depends on the decision made by the first reviewer.

The other aggregation operators can be interpreted in a similar manner.

The complete list of decision-oriented aggregation operators on **3** can naturally induce some new multigranulation rough set models, as we will show below.

Proposition 4. *Suppose that $(U, \mathbf{E})$ is a multigranulation space with $\mathbf{E} = \{R_1, R_2\}$, then for $X \subseteq U$, we have*

(1) $(\underline{apr}_{\mathbf{E}}^{d1}(X), \overline{apr}_{\mathbf{E}}^{d1}(X)) = (\underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X))$,

(2) $(\underline{apr}_{\mathbf{E}}^{d2}(X), \overline{apr}_{\mathbf{E}}^{d2}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X))$,

(3) $(\underline{apr}_{\mathbf{E}}^{d3}(X), \overline{apr}_{\mathbf{E}}^{d3}(X)) = (\underline{apr}_{R_1}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X))$,

(4) $(\underline{apr}_{\mathbf{E}}^{d4}(X), \overline{apr}_{\mathbf{E}}^{d4}(X)) = (\underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X))$,

(5) $(\underline{apr}_{\mathbf{E}}^{d5}(X), \overline{apr}_{\mathbf{E}}^{d5}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X))$,

(6) $(\underline{apr}_{\mathbf{E}}^{d6}(X), \overline{apr}_{\mathbf{E}}^{d6}(X)) = (\underline{apr}_{R_1}(X), \overline{apr}_{R_1}(X))$,

(7) $(\underline{apr}_{\mathbf{E}}^{d7}(X), \overline{apr}_{\mathbf{E}}^{d7}(X)) = (\underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X))$,

(8) $(\underline{apr}_{\mathbf{E}}^{d8}(X), \overline{apr}_{\mathbf{E}}^{d8}(X)) = (\underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X))$,

(9) $(\underline{apr}_{\mathbf{E}}^{d9}(X), \overline{apr}_{\mathbf{E}}^{d9}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \overline{apr}_{R_2}(X))$,

(10) $(\underline{apr}_{\mathbf{E}}^{d10}(X), \overline{apr}_{\mathbf{E}}^{d10}(X)) = (\underline{apr}_{R_1}(X), \overline{apr}_{R_2}(X))$,

(11) $(\underline{apr}_{\mathbf{E}}^{d11}(X), \overline{apr}_{\mathbf{E}}^{d11}(X)) = (\underline{apr}_{R_2}(X), \overline{apr}_{R_2}(X))$,

(12) $(\underline{apr}_{\mathbf{E}}^{d12}(X), \overline{apr}_{\mathbf{E}}^{d12}(X)) = (\underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X), \overline{apr}_{R_2}(X))$,

(13) $(\underline{apr}_{\mathbf{E}}^{d13}(X), \overline{apr}_{\mathbf{E}}^{d13}(X)) = (\underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(14) $(\underline{apr}_{\mathbf{E}}^{d14}(X), \overline{apr}_{\mathbf{E}}^{d14}(X)) = (\underline{apr}_{R_1}(X), \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(15) $(\underline{apr}_{\mathbf{E}}^{d15}(X), \overline{apr}_{\mathbf{E}}^{d15}(X)) = (\underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(16) $(\underline{apr}_{\mathbf{E}}^{d16}(X), \overline{apr}_{\mathbf{E}}^{d16}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$.

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Proof. It can be shown in a similar manner as that of Proposition 3.

Observe from Proposition 3 and Proposition 4 that

$$
\begin{aligned}
(\underline{apr}^{1}_{\mathbf{E}}(X), \overline{apr}^{1}_{\mathbf{E}}(X)) &= (\underline{apr}^{d2}_{\mathbf{E}}(X), \overline{apr}^{d2}_{\mathbf{E}}(X)),\\
(\underline{apr}^{2}_{\mathbf{E}}(X), \overline{apr}^{2}_{\mathbf{E}}(X)) &= (\underline{apr}^{d5}_{\mathbf{E}}(X), \overline{apr}^{d5}_{\mathbf{E}}(X)),\\
(\underline{apr}^{3}_{\mathbf{E}}(X), \overline{apr}^{3}_{\mathbf{E}}(X)) &= (\underline{apr}^{d3}_{\mathbf{E}}(X), \overline{apr}^{d3}_{\mathbf{E}}(X)),\\
(\underline{apr}^{4}_{\mathbf{E}}(X), \overline{apr}^{4}_{\mathbf{E}}(X)) &= (\underline{apr}^{d4}_{\mathbf{E}}(X), \overline{apr}^{d4}_{\mathbf{E}}(X)).
\end{aligned}
$$

However, the other multigranulation rough set models are different from those in Proposition 3. We can also observe that the multigranulation rough set models defined by the first decision-oriented aggregation operator is the optimistic multigranulation rough set model in [19] while the one defined by the sixteenth aggregation operator is the pessimistic multigranulation rough set model proposed in [19]. That is, the multigranulation rough set models proposed in [19] are just special cases of those induced by decision-oriented aggregation operators on **3**. □

3.2. Comments on the existing multigranulation rough set models

In this subsection, a comparative analysis of the existing multigranulation rough set models is performed. It is shown that they can be put into our proposed framework from the viewpoint of three-valued logic. Or in other words, they can be induced by the decision-oriented aggregation operators on **3**.

Note first that in the definition of multigranulation rough set models proposed in subsection 2.1, the aggregated value of 1 and 0 plays no role. This is also true for those multigranulation rough set models induced by some selected classes of binary relations such as those defined by reflexive relation, tolerance relation, pre-order relation. But for any binary relation without any constraint, the results in Proposition 4 should be modified as follows. Here we use $(\underline{apr}^{ai}_{\mathbf{E}}(X), \overline{apr}^{ai}_{\mathbf{E}}(X))$ to denote the ith pair of multigranulation approximation operators, with the superscript a indicating arbitrariness.

Proposition 5. *Suppose that $(U, \mathbf{E})$ is a multigranulation space with $\mathbf{E} = \{R_1, R_2\}$, then for $X \subseteq U$, we have*

(1) $(\underline{apr}^{a1}_{\mathbf{E}}(X), \overline{apr}^{a1}_{\mathbf{E}}(X)) = ((\underline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X)) \cup (\underline{apr}_{R_2}(X) \cap \overline{apr}_{R_1}(X)), (\overline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X)) \cap (\underline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X)))$,

(2) $(\underline{apr}^{a2}_{\mathbf{E}}(X), \overline{apr}^{a2}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), (\overline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X)) \cap (\underline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X)))$,

(3) $(\underline{apr}^{a3}_{\mathbf{E}}(X), \overline{apr}^{a3}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X), (\overline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X)) \cap (\underline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X)))$,

(4) $(\underline{apr}^{a4}_{\mathbf{E}}(X), \overline{apr}^{a4}_{\mathbf{E}}(X)) = (\overline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), (\overline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X)) \cap (\underline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X)))$,

(5) $(\underline{apr}^{a5}_{\mathbf{E}}(X), \overline{apr}^{a5}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X))$,

(6) $(\underline{apr}^{a6}_{\mathbf{E}}(X), \overline{apr}^{a6}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X))$,

(7) $(\underline{apr}^{a7}_{\mathbf{E}}(X), \overline{apr}^{a7}_{\mathbf{E}}(X)) = (\overline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X))$,

(8) $(\underline{apr}^{a8}_{\mathbf{E}}(X), \overline{apr}^{a8}_{\mathbf{E}}(X)) = ((\underline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X)) \cup (\underline{apr}_{R_2}(X) \cap \overline{apr}_{R_1}(X)), \overline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X))$,

(9) $(\underline{apr}^{a9}_{\mathbf{E}}(X), \overline{apr}^{a9}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \underline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(10) $(\underline{apr}^{a10}_{\mathbf{E}}(X), \overline{apr}^{a10}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X), \underline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(11) $(\underline{apr}^{a11}_{\mathbf{E}}(X), \overline{apr}^{a11}_{\mathbf{E}}(X)) = (\underline{apr}_{R_2}(X) \cap \overline{apr}_{R_1}(X), \underline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(12) $(\underline{apr}^{a12}_{\mathbf{E}}(X), \overline{apr}^{a12}_{\mathbf{E}}(X)) = ((\underline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X)) \cup (\underline{apr}_{R_2}(X) \cap \overline{apr}_{R_1}(X)), \underline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(13) $(\underline{apr}^{a13}_{\mathbf{E}}(X), \overline{apr}^{a13}_{\mathbf{E}}(X)) = ((\underline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X)) \cup (\underline{apr}_{R_2}(X) \cap \overline{apr}_{R_1}(X)), \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(14) $(\underline{apr}^{a14}_{\mathbf{E}}(X), \overline{apr}^{a14}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \overline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(15) $(\underline{apr}^{a15}_{\mathbf{E}}(X), \overline{apr}^{a15}_{\mathbf{E}}(X)) = (\underline{apr}_{R_2}(X) \cap \overline{apr}_{R_1}(X), \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$,

(16) $(\underline{apr}^{a16}_{\mathbf{E}}(X), \overline{apr}^{a16}_{\mathbf{E}}(X)) = (\underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X))$.

3.2.1. Comparative analysis with general framework for rough sets in multigranulation spaces

Recall that in [28], Yao and She proposed a unified framework for rough set models in multigranulation spaces. According to combination strategies used in the existing studies, two models are suggested. One model, called Model R, is based on a combination of a family of equivalence relations into an equivalence relation and the construction of approximations with respect to the combined relation. The other model, called Model A, is based on the construction of a family of approximations from a set of equivalence relations and a combination of the family of approximations.

Comparatively, our approach share common features with the approach presented in [28]. Precisely, these models are all based on the construction of a family of approximations with respect to a family of equivalence relations and the combination of these approximations. In this sense, the models proposed in our framework belong to model R in [28]. However, there exist visible difference between them. More precisely, in [28], the lower approximation and the upper approximation are dual to each other, while in our framework, the condition of duality has been withdrawn. Consequently, our results are more general than that in [28]. Moreover, our framework is presented from the viewpoint of three-valued logic while models in [28] are given from the standpoint of information fusion.

3.2.2. Comparative analysis with $(\cap, \cap)$ model and $(\cup, \cup)$ model

In [32], the authors investigated the general generation rules of approximation operators from the viewpoint of the union and intersection operations of rough approximation pairs. Four kinds of constructive methods of rough approximation operators from existing rough sets were established. The new notions of non-dual multigranulation rough sets and hybrid multi-granulation rough sets were introduced, and some properties were investigated. A trivial verification shows that the proposed $(\cap, \cap)$ model and $(\cup, \cup)$ model are the models induced by the second and the thirteenth decision-oriented aggregation operators in our framework, respectively.

3.2.3. Comparative analysis with rough set model in a multi-scale space

In [24,25], a formal approach to granular computing with multi-scale data measured at different levels of granulations was proposed. In that approach, the set of equivalence relations derived from different labels of scales forms a chain under the usual set-theoretic inclusion. That is, any two equivalence relations are comparable w.r.t. set-theoretical inclusion order. We call such special type of multigranulation space a multi-scale space. Let $(U, \{R_1, R_2\})$ be a multi-scale space satisfying the condition $R_1 \subseteq R_2$. Then for $X \subseteq U$, by applying set intersection and union, respectively, to the lower approximations in each Pawlak space, we obtain the lower approximation of X in the multi-scale space. Then define the upper approximation through duality. Since $\underline{apr}_{R_2}(X) \subseteq \underline{apr}_{R_1}(X) \subseteq X \subseteq \overline{apr}_{R_1}(X) \subseteq \overline{apr}_{R_2}(X)$, we have

$$\underline{apr}_{R_1}(X) \cup \underline{apr}_{R_2}(X) = \underline{apr}_{R_1}(X), \underline{apr}_{R_1}(X) \cap \underline{apr}_{R_2}(X) = \underline{apr}_{R_2}(X),$$
$$\overline{apr}_{R_1}(X) \cup \overline{apr}_{R_2}(X) = \overline{apr}_{R_2}(X), \overline{apr}_{R_1}(X) \cap \overline{apr}_{R_1}(X) = \overline{apr}_{R_1}(X),$$

consequently, two types of approximation pairs in a multi-scale table can be obtained, i.e., $(\underline{apr}_{R_1}(X), \overline{apr}_{R_1}(X))$, $(\overline{apr}_{R_2}(X), \overline{apr}_{R_2}(X))$. Moreover, if the condition of duality between lower approximation and upper approximation is withdrawn, we can also get two pairs of hybrid approximations, that is, $(\underline{apr}_{R_1}(X), \overline{apr}_{R_2}(X))$, $(\underline{apr}_{R_2}(X), \overline{apr}_{R_1}(X))$.

Returning to the multigranulation rough set models in Proposition 4, we can observe that four different types of multigranulation rough set models derived from a multi-scale space are indeed those defined through $*_6, *_7, *_{10}, *_{11}$, respectively. However, the aggregation operators corresponding to these four models are different from those given in Table 4. Precisely, for $R_1 \subseteq R_2$ and $x \in U$, $x \in \underline{apr}_{R_2}(X)$ implies that $x \in \underline{apr}_{R_1}(X)$. That is, both $0 * 1$ and $\frac{1}{2} * 1$ are not definable.

3.3. Characterization of three-way decision using three-valued logic

We now turn our attention to more general theory of three-way decision. We will consider the following scenario: there are two agents (agent 1 and agent 2), they will make decision for the same set of objects. We focus on how to represent the combined decision of an object from the viewpoint of three-valued logic.

According to the theory of three-way decision, for an object in the universe, each agent will make acceptance, rejection or noncommitment according to the values of evaluation function they use. Consequently, a ternary classification of the set of objects can be obtained, that is, the positive region, negative region and boundary region. We use P and N to denote the positive region and the negative region, respectively. Sometimes, we also call the pair (P, N) the decision region of an agent.

In what follows, we will consider the following two cases, depending on whether two agents have consistent decision or not.

Definition 6. Let (P_1, N_1) and (P_2, N_2) be the decision regions of agent 1 and agent 2, respectively. If $P_1 \cap N_2 = \emptyset$, $P_2 \cap N_1 = \emptyset$, then we say that two agents have consistent decisions, or briefly, two agents are consistent. Otherwise, we say that two agents are inconsistent.

According to Definition 6, two agents are consistent if and only if they do not make the conflicting decision. That is, for the same object, one agent accepts it while the other agent rejects it.

In what follows, we view decision combination as a mechanism by which agents can adapt their decisions in order to achieve a shared position or viewpoint. From this perspective we would expect a valid aggregation operator to generate a new decision which is always consistent with both the original decisions.

Taking this minimal requirement into account, we will define some combination operators in the following discussion, in a similar way to that in [9].

Definition 7. Conservative aggregation operator: Given two agents with associated decision region pair (P_1, N_1) and (P_2, N_2), we define the conservative aggregation operator such that:

$$(P_1, N_1) \otimes (P_2, N_2) = (P_1 \cap P_2, N_1 \cap N_2).$$

The term "conservative" refers to the fact that for an object, the combined decision is acceptance only if both agents accept it, and the combined decision is rejection only if both agents reject it. Trivially, such a combination corresponds to the sixteenth decision-oriented aggregation operator on **3** in Table 4.

Definition 8. Optimistic aggregation operator: Given two agents with associated decision region pair (P_1, N_1) and (P_2, N_2), we define the conservative aggregation operator such that:

$$(P_1, N_1) \oplus (P_2, N_2) = (P_1 \cup P_2, N_1 \cup N_2).$$

When two agents are inconsistent, the combined decision has conflicting information. That is, there exists at least an object which we accept and reject simultaneously. When two agents are consistent, it can be trivially checked that the combined decision is self-consistent. That is, $(P_1 \cup P_2) \cap (N_1 \cup N_2) = \emptyset$. Moreover, the combined decision pair is consistent with the original decision pairs. We can also conclude that such a aggregation operator of decision pairs corresponds to the first decision-oriented aggregation operator on **3** in Table 4. Optimistic multigranulation rough set model can be viewed as a concrete form of such a aggregation operator.

Definition 9. The Difference Operator: Given two agents with associated decision region pair (P_1, N_1) and (P_2, N_2), we define the difference operator such that:

$$(P_1, N_1) \ominus (P_2, N_2) = (P_1 \setminus N_2, N_1 \setminus P_2).$$

According to Definition 9, the positive region in the combined decision is obtained by removing those objects from the positive region of agent 1, which are rejected by agent 2. Similarly, the negative region is obtained by removing those objects from the negative region of agent 1, which are accepted by agent 2. Trivially, such a aggregation operator corresponds to the sixth decision-oriented aggregation operator on **3** in Table 4. When two agents are consistent, i.e., $P_1 \cap N_2 = \emptyset$, $P_2 \cap N_1 = \emptyset$, then one can check that $(P_1, N_1) \ominus (P_2, N_2) = (P_1, N_1)$.

Definition 10. The Consensus Operator: Given two agents with associated decision region pairs (P_1, N_1) and (P_2, N_2), we define the consensus operator such that:

$$(P_1, N_1) \odot (P_2, N_2) = ((P_1 \cup P_2) \setminus (N_1 \cup N_2), (N_1 \cup N_2) \setminus (P_1 \cup P_2)).$$

Consensus operator can be equivalently written as $(P_1, N_1) \odot (P_2, N_2) = ((P_1 \setminus N_2) \cup (P_2 \setminus N_1), (N_1 \setminus P_2) \cup (N_2 \setminus P_1))$. That is to say, consensus operator is obtained by firstly applying difference operator and then by using optimistic aggregation operator. An easy verification shows that if (P_1, N_1) and (P_2, N_2) are consistent, then $(P_1, N_1) \odot (P_2, N_2) = (P_1 \cup P_2, N_1 \cup N_2)$.

4. Five-valued logic semantics for multigranulation rough sets

In this section, we explore the idea of describing multigranulation rough sets using five-valued logic, whereby the value 1 corresponds to the certain positive region (see Definition 11) of a set, the value $\frac{3}{4}$ to the possible positive region (see Definition 11), the value $\frac{2}{4}$ to the certain boundary region (see Definition 11), the value $\frac{1}{4}$ to the possible negative region (see Definition 11), and the value 0 to the certain negative region (see Definition 11). Due to the properties of the above regions in multigranulation rough set theory, the semantics of the logic is described using a non-deterministic matrix (Nmatrix) [2].

4.1. Motivation for five-valued logic

For any set $X \subseteq U$, we can associate the five following regions in U, representing five basic statuses, or degrees, of membership of an object of the universe U in the set $X \subseteq U$:

Definition 11 *([1])*. Let $(U, \{R_i\}_{i\in N})$ be a multigranulation space and $X \subseteq U$, then x is said to be a
certain positive element of X, if $x \in \bigcap_{i\in N} \underline{apr}_{R_i}(X)$,
possible positive element of X, if $x \in \bigcup_{i\in N} \underline{apr}_{R_i}(X) \setminus \bigcap_{i\in N} \underline{apr}_{R_i}(X)$,
certain boundary element of X, if $x \in \bigcap_{i\in N} \overline{apr}_{R_i}(X) \setminus \bigcup_{i\in N} \underline{apr}_{R_i}(X)$,
possible negative element of X, if $x \in \bigcup_{i\in N} \overline{apr}_{R_i}(X) \setminus \bigcap_{i\in N} \overline{apr}_{R_i}(X)$, and
certain negative element of X, if $x \in (\bigcup_{i\in N} \overline{apr}_{R_i}(X))^c$.
The sets of certain positive elements, possible positive elements, certain boundary, possible negative and certain negative elements of X are denoted by $CPos(X), PPos(X), CBnd(X), PNeg(X), CNeg(X)$, respectively.

This suggests a natural way of describing rough sets with help of a simple five-valued logic L_{mrs}, defined informally as follows:

The formulas of L_{mrs} are all expressions of the form Ax, where A is an expression representing a subset of U and x is a variable representing an object in U.

The semantics of L_{mrs} uses logical values in $T = \{1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}, 0\}$ where:

1 represents the classical value true,
$\frac{3}{4}$ represents the non-classical value possibly true,
$\frac{2}{4}$ represents a non-classical value unknown,
$\frac{1}{4}$ represents a non-classical value possibly false,
0 represents a classical value false.

The truth-values of formulas in L_{mrs} with respect to a multigranulation space $K = (U, \{R_i\}_{i\in N})$,an interpretation $|\cdot|$ of set expressions, and a valuation v of object variables, are as follows:

$$\| Ax \|_v = \begin{cases} 1, & v(x) \in CPos(v(A)), \\ \frac{3}{4}, & v(x) \in PPos(v(A)), \\ \frac{2}{4}, & v(x) \in CBnd(v(A)), \\ \frac{1}{4}, & v(x) \in PNeg(v(A)), \\ 0, & v(x) \in CNeg(v(A)). \end{cases} \tag{14}$$

4.2. Motivation for the use of non-deterministic matrices

Owing to the fact that four types of multigranulation approximation operators obey the following rules

$$\bigcap_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X^c) = (\bigcup_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X))^c,$$
$$\bigcup_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X^c) = (\bigcap_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X))^c,$$
$$\bigcap_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X\cap Y) = \bigcap_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X) \cap \bigcap_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(Y),$$
$$\bigcup_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X\cap Y) \subseteq \bigcup_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X) \cap \bigcup_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(Y),$$
$$\bigcup_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X\cup Y) = \bigcup_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X) \cup \bigcup_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(Y),$$
$$\bigcap_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X\cup Y) \supseteq \bigcap_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X) \cup \bigcap_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(Y),$$
$$\bigcup_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X^c) = (\bigcap_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X))^c,$$
$$\bigcap_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X^c) = (\bigcup_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X))^c,$$
$$\bigcup_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X\cap Y) \subseteq \bigcup_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X) \cap \bigcup_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(Y),$$
$$\bigcap_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X\cap Y) \subseteq \bigcap_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X) \cap \bigcap_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(Y),$$
$$\bigcap_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X\cup Y) \supseteq \bigcap_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(X) \cup \bigcap_{E\in\{R_i\}_{i\in N}} \overline{apr}_E(Y),$$
$$\bigcup_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X\cup Y) \supseteq \bigcup_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(X) \cup \bigcup_{E\in\{R_i\}_{i\in N}} \underline{apr}_E(Y),$$

the semantics of the proposed logic L_{mrs} for multigranulation rough set is not decompositional. More precisely, the values of $(A \vee B)x$ and $(A \wedge B)x$ are not always uniquely determined by the values of Ax and Bx. We cannot in general say in advance which value in $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ will be assigned to $(A \vee B)x$ (or $(A \wedge B)x$) by the interpretation to the formulas Ax and Bx.

Owing to the above fact, the semantics of L_{rs} cannot be defined using an ordinary logical matrix. A solution to this problem is to use instead a non-deterministic logical matrix (Nmatrix), which is a generalization of an ordinary matrix modeling non-determinism, with interpretations of logical connectives returning sets of logical values instead of single values.

Table 5
Semantics for $\tilde{\neg}$.

$\tilde{\neg}$	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	0
	1	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

Table 6
Semantics for $\tilde{\vee}$.

$\tilde{\vee}$	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1
0	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1
$\frac{1}{4}$	$\frac{1}{4}$	$\{\frac{1}{4}, \frac{2}{4}\}$	$\{\frac{2}{4}, \frac{3}{4}\}$	$\{\frac{3}{4}, 1\}$	1
$\frac{2}{4}$	$\frac{2}{4}$	$\{\frac{2}{4}, \frac{3}{4}\}$	$\{\frac{2}{4}, \frac{3}{4}, 1\}$	$\{\frac{3}{4}, 1\}$	1
$\frac{3}{4}$	$\frac{3}{4}$	$\{\frac{3}{4}, 1\}$	$\{\frac{3}{4}, 1\}$	$\{\frac{3}{4}, 1\}$	1
1	1	1	1	1	1

Table 7
Semantics for $\tilde{\wedge}$.

$\tilde{\wedge}$	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	1
0	0	0	0	0	0
$\frac{1}{4}$	0	$\{0, \frac{1}{4}\}$	$\frac{1}{4}$	$\{0, \frac{1}{4}\}$	1
$\frac{2}{4}$	0	$\frac{1}{4}$	$\{0, \frac{1}{4}, \frac{2}{4}\}$	$\{\frac{1}{4}, \frac{2}{4}\}$	1
$\frac{3}{4}$	0	$\{0, \frac{1}{4}\}$	$\{\frac{1}{4}, \frac{2}{4}\}$	$\{\frac{2}{4}, \frac{3}{4}\}$	1
1	0	1	1	1	1

Definition 12 *([2])*. A non-deterministic matrix (Nmatrix) for a propositional language L is a tuple $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where:

- $\mathcal{T}$ is a non-empty set of truth values.
- $\emptyset \subset \mathcal{D} \subseteq \mathcal{T}$ is the set of designated values.
- For every n-ary connective $\diamond$ of L, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{\diamond}$ from $\mathcal{T}^n$ to $2^{\mathcal{T}} \setminus \{\emptyset\}$.

Let W be the set of well-formed formulas of L. A (legal) valuation in an Nmatrix $\mathcal{M}$ is a function $v : W \to \mathcal{T}$ that satisfies the following condition:

$$v(\diamond(\psi_1, \cdots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \cdots, v(\psi_n))$$

for every n-ary connective $\diamond$ of L and any $\psi_1, \cdots, \psi_n \in W$.

4.3. Logic language for multigranulation rough set

The predicate language L_{mrs} for describing multigranulation rough sets uses only unary predicate symbols representing sets, object variables, and the symbols $\neg, \vee, \wedge$ representing operations on predicates, i.e.,

$\mathbf{P}_1 = \{P, Q, R, \cdots\}$, $\mathbf{P}_n = \emptyset, n \geq 2$,
$\mathbf{O}_1 = \{-\}$, $\mathbf{O}_2 = \{\cup, \cap\}$, $\mathbf{O}_k = \emptyset, k \geq 3$,
where $\mathbf{P}_k, \mathbf{O}_k$ denote the set of k-ary predicate and k-ary connective.

Thus the set W of well-formed formulas of L_{mrs} contains all expressions of the form Ax, where A is a unary predicate expression representing a set, built of the predicate symbols in $\mathbf{P}_1$ and using the operation symbols $\neg, \vee, \wedge$ while $x \in V$ is an individual variable.

The semantics of L_{mrs} is given by the Nmatrix $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where $T = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$, $D = \{1\}$, and $\neg, \vee, \wedge$ are interpreted as set-theoretic operations on multigranulation rough sets. Their semantics are given in the following proposition.

Proposition 6. *(i) The semantic of $\tilde{\neg}$ is given in Table 5,*
(ii) The semantic of $\tilde{\vee}$ is given in Table 6,
(iii) The semantic of $\tilde{\wedge}$ is given in Table 7.

Proof. (i) It suffices to show that $\| Ax \|_v = 1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}, 0$ implies that $\| (\neg A)x \|_v = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$, respectively. We will consider separately the following cases:

- $\| Ax \|_v = 1$: According to (13), $\| Ax \|_v = 1$ means that $v(x) \in \bigcap_{i\in N} \underline{apr}_{R_i}(|A|)$, which together with $\bigcap_{i\in N} \underline{apr}_{R_i}(|A|) = (\bigcup_{i\in N} \overline{apr}_{R_i}(|A|^c))^c$ implies that $v(x) \in (\bigcup_{i\in N} \overline{apr}_{R_i}(|A|^c))^c$, that is, $v(x) \notin \bigcup_{i\in N} \overline{apr}_{R_i}(|\neg A|)$, or in other words, $v(x) \in CNeg(|\neg A|)$, and therefore, $\| (\neg A)x \|_v = 0$.
- $\| Ax \|_v = \frac{3}{4}$: According to (13), $\| Ax \| = \frac{3}{4}$ means that $v(x) \in \bigcup_{i\in N} \underline{apr}_{R_i}(|A|) \setminus \bigcap_{i\in N} \underline{apr}_{R_i}(|A|)$. Since $\bigcup_{i\in N} \underline{apr}_{R_i}(|\neg A|) \setminus \bigcap_{i\in N} \underline{apr}_{R_i}(|\neg A|) = \bigcup_{i\in N} \underline{apr}_{R_i}(|A|^c) \setminus \bigcap_{i\in N} \underline{apr}_{R_i}(|A|^c) = (\bigcap_{i\in N} \overline{apr}_{R_i}(|A|))^c \setminus (\bigcup_{i\in N} \overline{apr}_{R_i}(|A|))^c$, which is also equal to

$\bigcup_{i\in N}\overline{apr}_{R_i}(|A|)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(|A|)$, then we obtain $v(x)\in\bigcup_{i\in N}\overline{apr}_{R_i}(|A|)$ and $v(x)\notin\bigcap_{i\in N}\overline{apr}_{R_i}(|A|)$, and hence, $v(x)\in PNeg(|A|)$, i.e., $\|(\neg A)x\|=\frac{1}{4}$.

The others can be shown similarly.

We also give the proof of the semantics for $\tilde{\barwedge}$ by considering the following cases:

- $\frac{1}{4}\tilde{\barwedge}\frac{1}{4}=\{0,\frac{1}{4}\}$. According to the definition, $\|Ax\|_v=\frac{1}{4}$ means that

$$v(x)\in\bigcup_{i\in N}\overline{apr}_{R_i}(|A|)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(|A|). \tag{15}$$

Similarly, $\|Bx\|=\frac{1}{4}$ means that

$$v(x)\in\bigcup_{i\in N}\overline{apr}_{R_i}(|B|)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(|B|). \tag{16}$$

We can show that $v(x)\notin\bigcap_{i\in N}\overline{apr}_{R_i}(|A\wedge B|)=\bigcap_{i\in N}\overline{apr}_{R_i}(|A|\cap|B|)$ below. Indeed, suppose, on the contrary, that $v(x)\in\bigcap_{i\in N}\overline{apr}_{R_i}(|A|\cap|B|)$, we then have from the monotone property of $\bigcap_{i\in N}\overline{apr}_{R_i}$ that $v(x)\in\bigcap_{i\in N}\overline{apr}_{R_i}(|A|)$ and $v(x)\in\bigcap_{i\in N}\overline{apr}_{R_i}(|B|)$, which, however, contradict with (14) and (15). This leads to the conclusion that only two cases, i.e., $v(x)\in\bigcup_{i\in N}\overline{apr}_{R_i}\setminus\bigcap_{i\in N}\overline{apr}_{R_i}$ and $v(x)\notin\bigcup_{i\in N}\overline{apr}_{R_i}$, are possible, that is, $\|(A\wedge B)x\|\in\{0,\frac{1}{4}\}$. The existence of these two values is guaranteed by the following examples in the set-theoretic environment:

$\bullet_1$ $U=\{1,2,3,4,5,6\}$, $U/R_1=\{\{1,2\},\{3,4\},\{5,6\}\}$, $U/R_2=\{\{1,3\},\{2,5\},\{4,6\}\}$. Take $A=\{1,4\}$, $B=\{2,5\}$ and $x=6$, then a trivial computation shows that $x\in\bigcup_{i\in N}\overline{apr}_{R_i}(A)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(A)$ and $x\in\bigcup_{i\in N}\overline{apr}_{R_i}(B)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(B)$. However, since $A\cap B=\emptyset$, we have $x\in(\bigcup_{i\in N}\overline{apr}_{R_i}(A\cap B))^c$.

$\bullet_2$ $U=\{1,2,3,4,5,6\}$, $U/R_1=\{\{1,2\},\{3,4\},\{5,6\}\}$, $U/R_2=\{\{1,3,4\},\{2,5\},\{6\}\}$. Take $A=\{1,3\}$, $B=\{1\}$ and $x=2$, then a trivial computation shows that $x\in\bigcup_{i\in N}\overline{apr}_{R_i}(A)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(A)$ and $x\in\bigcup_{i\in N}\overline{apr}_{R_i}(B)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(B)$. Since $A\cap B=B$, we therefore have $x\in\bigcup_{i\in N}\overline{apr}_{R_i}(A\cap B)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(A\cap B)$.

$\bullet\bullet$ $\frac{1}{4}\tilde{\barwedge}\frac{2}{4}=\{0,\frac{1}{4}\}$. According to (13), $\|Ax\|_v=\frac{1}{4}$ means that

$$v(x)\in\bigcup_{i\in N}\overline{apr}_{R_i}(|A|)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(|A|). \tag{17}$$

Similarly, $\|Bx\|_v=\frac{2}{4}$ means that

$$v(x)\in\bigcap_{i\in N}\overline{apr}_{R_i}(|B|)\setminus\bigcup_{i\in N}\underline{apr}_{R_i}(|B|). \tag{18}$$

The detailed proof consists of the following steps:

(i) $v(x)\notin\bigcap_{i\in N}\overline{apr}_{R_i}(|A\wedge B|)=\bigcap_{i\in N}\overline{apr}_{R_i}(|A|\cap|B|)$. Suppose, on the contrary, that $v(x)\in\bigcap_{i\in N}\overline{apr}_{R_i}(|A|\cap|B|)$, then we have from the monotone property of multigranulation approximation operator that $v(x)\in\bigcap_{i\in N}\overline{apr}_{R_i}(|A|)$, which, however, contradicts with (17).

(ii) $v(x)\notin\bigcup_{i\in N}\underline{apr}_{R_i}(|A\wedge B|)\setminus\bigcap_{i\in N}\underline{apr}_{R_i}(|A|\cap|B|)$. This can be proved by using a similar manner as in (i).

(iii) $v(x)\notin\bigcap_{i\in N}\overline{apr}_{R_i}(|A\wedge B|)\setminus\bigcup_{i\in N}\underline{apr}_{R_i}(|A|\cap|B|)$. Suppose, on the contrary, that $v(x)\in\bigcap_{i\in N}\overline{apr}_{R_i}(|A\wedge B|)\setminus\bigcup_{i\in N}\underline{apr}_{R_i}(|A|\cap|B|)$, then it follows from the monotonicity of multigranulation approximation operator that $v(x)\notin\bigcap_{i\in N}\overline{apr}_{R_i}(|B|)$, which, however, contradicts with (18).

The above fact shows that $\|(A\wedge B)x\|_v\notin\{\frac{2}{4},\frac{3}{4},1\}$, that is, $\|(A\wedge B)x\|_v\in\{0,\frac{1}{4}\}$. Last but not the least, the existence of these two values is guaranteed by the following examples:

$\bullet\bullet_1$ $U=\{1,2,3,4,5,6\}$, $U/R_1=\{\{1,2\},\{3,4\},\{5,6\}\}$, $U/R_2=\{\{1,3\},\{2,5\},\{4,6\}\}$. Take $A=\{1,4\}$, $B=\{1,2\}$ and $x=3$, then a trivial computation shows that $x\in\bigcap_{i\in N}\overline{apr}_{R_i}(A)\setminus\bigcup_{i\in N}\underline{apr}_{R_i}(A)$ and $x\in\bigcup_{i\in N}\overline{apr}_{R_i}(B)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(B)$. However, since $A\cap B=\{1\}$, we have $x\in\bigcup_{i\in N}\overline{apr}_{R_i}(A\cap B)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(A\cap B)$.

$\bullet\bullet_2$ $U=\{1,2,3,4,5,6\}$, $U/R_1=\{\{1,2\},\{3,4\},\{5,6\}\}$, $U/R_2=\{\{1,3,6\},\{2,5\},\{4\}\}$. Take $A=\{1,4\}$, $B=\{2,6\}$ and $x=3$, then a trivial computation shows that $x\in\bigcap_{i\in N}\overline{apr}_{R_i}(A)\setminus\bigcup_{i\in N}\underline{apr}_{R_i}(A)$ and $x\in\bigcup_{i\in N}\overline{apr}_{R_i}(B)\setminus\bigcap_{i\in N}\overline{apr}_{R_i}(B)$. However, since $A\cap B=\emptyset$, we have $x\in(\bigcup_{i\in N}\overline{apr}_{R_i}(A\cap B))^c$.

Indeed, it cannot be difficult to check that the "determinization" of Tables 6 and 7 are the three basic t-norms and t-conorms: Gödel, Łukasiewicz and product. □

5. Concluding remarks

In this paper, a modest attempt is made to connect rough sets in multigranulation spaces to multiple-valued logics. By means of the existing conjunctions on **3**, various types of multigranulation rough set models have been derived. Moreover, we present an axiomatic definition of decision-oriented aggregation operators on **3** from the viewpoint of three-way decision. These aggregation operators correspond to altogether 16 types of multigranulation rough set models. Comparatively, these models have the advantage in embodying the idea of three-way decision. Furthermore, it is shown that the existing rough set models in multigranulation spaces can be put into the framework of our paper, which means that they can be

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interpreted from the viewpoint of three-valued logic. We also give a five-valued semantic for multigranulation rough set models.

These obtained results provide an analysis of the existing multigranulation rough set models, as well as generalize them from the viewpoint of three-valued logics. We hope that the obtained results can bring new insights into establishment of decision-oriented rough set models in the future. Moreover, as pointed out by one reviewer, the five-valued semantic in Section 4 should be explored further. Future work also includes sound and complete, cut-free sequent calculi for the logic L_{mrs} generated by the rough set Nmatrix.

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