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A cautious ranking methodology with its application for stock screening

Peng Song[a,b], Jiye Liang[b,c,*], Yuhua Qian[b,c], Wei Wei[b,c], Feng Wang[b,c]

[a] School of Economics and Management, Shanxi University, Taiyuan 030006, Shanxi, China
[b] Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Taiyuan 030006, Shanxi, China
[c] School of Computer and Information Technology, Shanxi University, Taiyuan 030006, Shanxi, China

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ABSTRACT

To reasonably and effectively solve the ranking decision problem, there currently exist various meaningful methods that are focused on different perspectives. However, the ranking decision problem is a systematic issue that involves data representation, the dominance relation, feature selection, and ranking mechanism. In this study, we aimed to build a novel ranking methodology by taking into account both the inherent multicriteria nature of practical decision situations and cautious decision makers' preferences. In order to better reveal the entirety of the data set, the form of interval data is introduced to characterize the ranges of attribute values. For the purpose of improving the decision performance, we develop a measurement called interval ordered conditional entropy to extract the most representative condition attributes having significant ordered relevance to the decision attribute. Based on the cautious dominance relation introduced for interval data, a two-step ranking mechanism with cautious characteristics is introduced that utilizes an interval ordered information table organized according to the previously selected informative attributes. In addition, the validity of this ranking method is tested through a detailed case study on stock screening decisions involving three successive rounds of tests. The corresponding results indicate the effectiveness of the methodological approach proposed in this paper.

1. Introduction

In practical decision environments, ranking decision problems, such as that involved in ranking universities [1,2], venture investment [3], and market segment evaluation [4], is a typical type of decision problem. Thus far, many effective and reasonable ranking methods have been constructed, including TOPSIS [5,6], ELECTRE [7], the analytic hierarchy process (AHP) [6,8], etc. It should be noted that, in the past decades, the forms of data representation appear to have become more complex because of the increasingly intricate and uncertain decision situations. Accordingly, many researchers have paid considerable attention to ranking approaches in the context of interval-valued data [3,9–11].

In realistic decision problems, it is increasingly found that the decision behaviors of decision makers act as important influence factors in their decision making. Yao built the theory of three-way decisions, which effectively characterized the common human behavior of trisecting a universal set in problem solving [12,13]. Wu and Chiclana proposed new attitudinal expected score and accuracy functions for ranking interval-valued intuitionistic fuzzy numbers that take into account the characteristics of decision makers' risk attitude [14]. Ruan and Shi integrated interval comparison techniques into scenario analysis methods for monitoring and assessing fruit freshness in an IOT-based e-commerce delivery system [15]. A review of the existing literature shows that the issue of ranking interval data considering decision behaviors has become a promising and interesting research field [14–18]. In reality, risk aversion is a typical type of human decision behavior [19,20]. In particular on the background of exacerbated market risk after the global financial crisis in 2008 [21], the study of risk aversion-based ranking models is of significance and value. Chen and Zou developed a new group decision-making method to solve the supplier selection problem, in which the weights of the decision makers are appropriate for risk avoiders under an uncertain environment [22]. Through introducing the hyperbolic absolute risk aversion utility function, Gao et al. built a new operator for group decision making that can reflect the decision makers' risk attitude [23]. The above methods

* Corresponding author at: Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Taiyuan 030006, Shanxi, China.

E-mail addresses: songpeng@sxu.edu.cn (P. Song), ljy@sxu.edu.cn (J. Liang), jinchengqyh@sxu.edu.cn (Y. Qian), weiwei@sxu.edu.cn (W. Wei), sxuwangfeng@126.com (F. Wang).

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effectively reflect the decision behavior of risk averse decision makers from their specific perspectives. However, the cautious ranking decision problem is a systematic issue. Therefore, a methodological approach that considers risk aversion behavior in each step of the problem solving (e.g., in the dominance relation, attribute weights, and ranking mechanism) is desirable in the context of interval data.

In general, special ranking decision issues have specific decision targets. In fact, in certain important and classical ranking decision problems, such as stock screening and venture investment, feature selection is a key preprocessing step of the ranking methodology [3,24]. Given the nature of the aforementioned decision problems, the basic issue is to discover the informative indicators that have significant ordered relevance to the stock return or investment return. Through using an effective and reasonable feature selection algorithm, we can obtain the pertinent evaluation attributes, and then effectively provide a ranking result for improving the investment performance [25]. Obviously, feature selection is an indispensable component in the overall ranking procedure in the type of decision situation mentioned above. It should be noted that attribute reduction in rough set theory, introduced by Pawlak [26,27], has become a useful tool that is widely used for feature selection [3,28–34]. Ziarko defined the notion of β-reduct and provided a battery of attribute reduction approaches in the variable precision rough set model [30]. To address the problem of time complexity in dynamic datasets, Chen et al. proposed an incremental algorithm for attribute reduction in the variable precision rough set model [31].

The research outcomes mentioned above provided a wide variety of effective feature selection algorithms for problem solving in different complex and uncertain situations. It should be emphasized that, to handle ranking decision problems with interval data effectively, Qian [3] presented an attribute reduction approach based on a discernibility matrix. In fact, discernibility matrix-based approaches constitute a typical type of attribute reduction method in rough set theory [35]. However, they have a disadvantage in that the discernibility function does not fully correspond with the specific decision target. Basically, given the nature of ranking decisions, feature selection should consider the ordered relevance between the key evaluation feature subsets and the decision target. Consequently, the exploration of the feature selection approach that takes into account the characteristic of ordered relevance has naturally become a relevant research issue.

In the area of practical decision situations, the issue of stock screening can be treated as a ranking decision problem [36,37] and many valuable multiple criteria decision making (MCDM) methods are also constantly emerging [7,38,39]. Xidonas et al. presented an ELECTRE Tri method, in which three types of investors' preferences, consisting of the conservative, balanced, and aggressive investment profiles, are considered in the decision-making process of stock selection [38]. Considering both the preferences involved in evaluating the relevant criteria and the decision behaviors of investors, Sevastjanov and Dymova proposed a cautious stock ranking strategy that considers the high level coincidence between a firm's financial performance and its market success in the context of fuzzy-valued data [37]. Indeed, the strategy is quite credible, because the proposed program can reject "unsafe" firms, the market success of which may be derived from the subjective experts' opinions, rumors, or other factors. However, two important problems related to this cautious strategy remain to be considered, one of which is the determination of an appropriate membership function of fuzzy subsets, which remains a challenging issue that needs to be addressed [36,37]. Indeed, in practical issues, interval data have become a significant type of data representation form for better revealing the entirety of a data set [40–42]. In particular for stock screening decisions, most indicators always show the characteristic of fluctuations. In fact, the form of interval data has a better capability to reflect this type of uncertainty [40,41]. Therefore, in this study it was considered desirable that the representation of interval data be introduced into the decision process. The second problem is the determination of the informative financial indicators that have significant ordered relevance to stock return in order to improve investment performance by means of using an effective and reasonable feature selection algorithm [25,43]. It is considered that feature selection is the key process in the stock screening decision [44]. Overall, the stock screening decision is a systematic issue that involves a series of successive problems, including data representation, feature selection, and ranking mechanism.

In summary, the objective of this study was to develop a novel ranking methodology that involves interval data representation, feature selection, and a ranking mechanism in the context of a risk averse situation. This was the motivation of our research. In this study, our objective was to solve three key problems:

- How can a systematic cautious ranking method for interval data be constructed?
- How can an ordered-relevance-preserving feature selection method in the proposed cautious ranking methodology be built?
- How can stock screening decision problems be solved by employing the proposed cautious ranking methodological approach?

To build a novel cautious ranking methodological approach, in this study, we first introduced a cautious dominance relation into the initial decision table as the fundamental representation for depicting ordered information. Then, on the basis of interval data transformation for all the attributes and a subsequent sorting process for the decision attributes, we were able develop a new interval ordered decision table (IODT). Based on the previous steps, a measurement of ordered relevance, called interval ordered conditional entropy, for extracting the representative feature subsets was devised. Finally, in terms of the above selected pertinent features, we introduce a cautious ranking mechanism to obtain the ranking results. Simultaneously, in the process of decision modeling analysis, the validity of the proposed approach was verified by a case study on stock screening. The proposed methodology can further improve the investment performance of stock screening decisions. The flow chart of cautious ranking methodology is shown in Fig. 1.

The remainder of this paper is organized as follows. In Section 2, we review some preliminary notions and definitions related to the dominance relation, information entropy in rough set theory, and ranking mechanisms. In Section 3, we describe the building of a systematic cautious ranking methodology that includes a sequence of decision processes, namely, a cautious dominance relation for interval data, interval data-based data representation, a feature selection method, and a two-step ranking mechanism. In Section 4, we present a detailed case study of stock screening decisions to verify the effectiveness of the methodology proposed in this paper. Finally, Section 5 concludes this paper by presenting some remarks and discussions.

2. Preliminaries

An *information system* (IS) is a quadruple $S=(U, AT, V, f)$, where U denotes a finite non-empty set of objects and AT an attribute set, $V=\bigcup_{a\in AT}V_a$ and V_a represents a domain of attribute a, and $f: U\times AT\to V$ is a total function such that $f(x, a)\in V_a(a\in AT, x\in U)$, namely, an information function [27,45]. An IS is called an *interval information system* (IIS) if V_a is a set of interval-valued numbers, where $f(x, a)$ is denoted by

$$f(x,a)=[a^L(x),a^U(x)]=\{p\mid a^L(x)\leqslant p\leqslant a^U(x),a^L(x),a^U(x)\in \mathbf{R}\}.$$

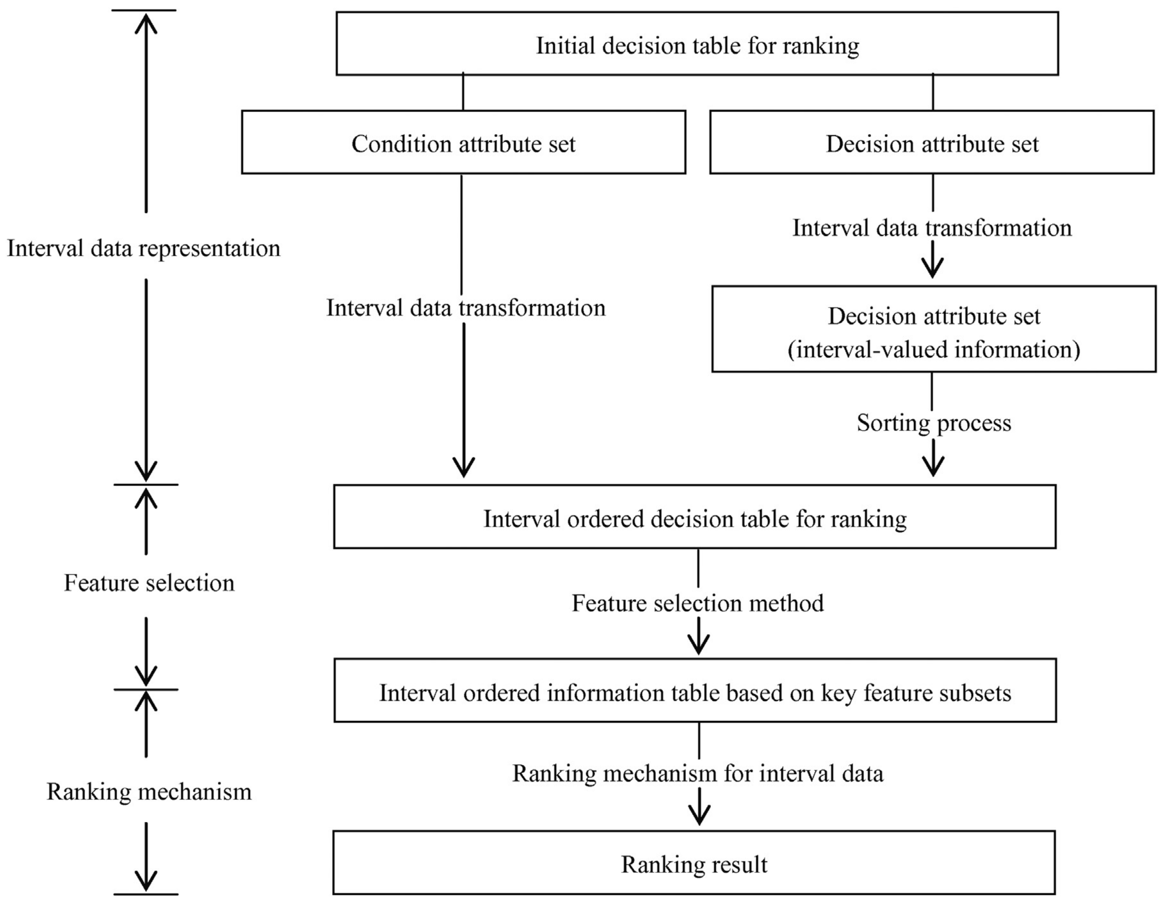


Fig. 1. Flow chart of cautious ranking methodology.

In particular, a single-valued IS can be regarded as a special form of IIS, in which $f(x, a) = a^L(x) = a^U(x)$. Furthermore, an IIS is called an *interval ordered information system* (IOIS) when all attributes are criteria [3], which can usually be categorized into two classes, namely, profit type and cost type.

2.1. Dominance relation

In fact, the dominance relation is an important basis for the ranking task. Here, we give several different definitions of the dominance relation in IOISs.

Definition 2.1. Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. The *dominance relation* with respect to A can be defined as [3]

(1) $R_A^{\geqslant_1} = \{(y, x) \in U \times U \mid a_1^L(y) \geqslant a_1^L(x)(\forall a_1 \in A_1);\ \ a_2^L(y) \leqslant a_2^L(x)(\forall a_2 \in A_2)\}$, or
(2) $R_A^{\geqslant_2} = \{(y, x) \in U \times U \mid a_1^U(y) \geqslant a_1^U(x)(\forall a_1 \in A_1);\ \ a_2^U(y) \leqslant a_2^U(x)(\forall a_2 \in A_2)\}$, or
(3) $R_A^{\geqslant_3} = \{(y, x) \in U \times U \mid a_1^L(y) \geqslant a_1^U(x)(\forall a_1 \in A_1);\ \ a_2^U(y) \leqslant a_2^L(x)(\forall a_2 \in A_2)\}$.

In view of the definition of the dominance relation, we say that y dominates x with respect to the attribute set A denoted by $y \succcurlyeq_A x$ if $yR_A^{\geqslant}x$. Furthermore, from the definition of $R_A^{\geqslant_1}$, the set of objects dominating x, called the dominance class $[x]_A^{\geqslant_1}$, can also be induced, i.e.,

$$[x]_A^{\geqslant_1} = \{y \in U \mid a_1^L(y) \geqslant a_1^L(x)(\forall a_1 \in A_1);\ \ a_2^L(y) \leqslant a_2^L(x)(\forall a_2 \in A_2)\} = \{y \in U \mid (y, x) \in R_A^{\geqslant_1}\}.$$

Similarly, the other formulations of a dominance class can be easily provided based on the corresponding definitions of the dominance relation. In practice, different dominance relations have different practical implications and are respectively applicable to specific ranking decision targets. Here, we use $R_A^{\geqslant}$ and $[x]_A^{\geqslant}$ to respectively represent the general formations of the dominance relation and the dominance class.

2.2. Information entropy in rough set theory

According to the inherent idea of feature selection in ranking methodology, ordered relevance refers to ordered consistency in nature. Ordered consistency means that object y should be preferred to object x with respect to the decision attribute if its values are superior to those of object x under the condition attributes; otherwise, the term used is ordered inconsistency. Consequently, both consistency and inconsistency constitute the uncertainty of an ordered decision.

In the measurement of uncertainty, information entropy, introduced by Shannon [46], has played an increasingly important role. In particular for ISs, information entropy, which is extended in rough set theory, has become an important tool for feature selection, because it can effectively characterize the information content [28,32,33,47].

To characterize the information content in IOISs, the interval ordered information entropy is presented as follows.

Definition 2.2. Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$, $U/R_A^{\geqslant} = \{[x_1]_A^{\geqslant}, [x_2]_A^{\geqslant}, \ldots, [x_{|U|}]_A^{\geqslant}\}$; then, the interval ordered information entropy of $A^{\geqslant}$ is defined as [48]

$$E(A^{\geqslant}) = \sum_{i=1}^{|U|} \frac{1}{|U|} \left(1 - \frac{|[x_i]_A^{\geqslant}|}{|U|}\right).$$

Accordingly, the interval ordered mutual information between $A^{\geqslant}$ and $B^{\geqslant}$ in an IOIS can be defined by [10]

$$E(A^{\geqslant}; B^{\geqslant}) = \sum_{i=1}^{|U|} \frac{1}{|U|} \frac{|[x_i]_A^{\geqslant c} \cap [x_i]_B^{\geqslant c}|}{|U|},$$

where $[x_i]_A^{\geqslant c} = U - [x_i]_A^{\geqslant}$, and $[x_i]_B^{\geqslant c} = U - [x_i]_B^{\geqslant}$.

2.3. Ranking mechanisms based on dominance relation

The ranking mechanism is the pivotal step in the overall ranking methodology. On the basis of the dominance relation, the corresponding measurement, called the dominance degree, for ranking objects in IOISs is as follows.

Definition 2.3. Let $S=(U, AT, V, f)$ be an IOIS and $A \subseteq AT$. The *dominance degree* between two objects in terms of the dominance relation $R_A^{\geqslant}$ is defined as [3]

$$D_A(x_i, x_j) = \frac{|[x_i]_A^{\geqslant c} \cup [x_j]_A^{\geqslant}|}{|U|}.$$

According to the concept of dominance degree, we can measure the preferability degree of one object over the second object as regards the attribute set A. Thus, a comprehensive score of one object, namely, the entire dominance degree, can be defined as

$$D_A(x_i) = \frac{1}{|U|-1} \sum_{j \neq i} D_A(x_i, x_j), \quad x_i, \quad x_j \in U.$$

Indeed, the dominance degree primarily describes the relative ranking position of alternatives. To describe the more detailed difference between the attribute values of alternatives, a measurement called the directional distance index has been proposed.

Definition 2.4. Given $f(x_i, a)=[a^L(x_i), a^U(x_i)]$ and $f(x_j, a)=[a^L(x_j), a^U(x_j)]$, the *directional distance index* between two objects with respect to attribute a is defined as [10]

$$DDI_a(x_i, x_j) = \frac{1}{2} + \frac{1}{4} \frac{a^U(x_i) - a^U(x_j) + a^L(x_i) - a^L(x_j)}{\max(a^U(x)) - \min(a^L(x))},$$

where $max(a^U(x)) = max\{a^U(x_1), a^U(x_2), \ldots, a^U(x_{|U|})\}$, $min(a^L(x)) = min\{a^L(x_1), a^L(x_2), \ldots, a^L(x_{|U|})\}$, and $x_i, x_j \in U$. In particular, $DDI_a(x_i, x_j) = \frac{1}{2}$ if $max(a^U(x)) = min(a^L(x))$.

Furthermore, the corresponding measurement with respect to attribute set A is defined by

$$DDI_A(x_i, x_j) = \sum_{\forall a \in A} w_a \cdot DDI_a(x_i, x_j),$$

where $w_a = \frac{E(a^{\geqslant}; A^{\geqslant})}{\sum_{\forall a \in A} E(a^{\geqslant}; A^{\geqslant})}$ represents the weight of attribute a.

Similarly to the definition of entire dominance degree, the formulation of the entire directional distance index can be represented as

$$DDI_A(x_i) = \frac{1}{|U|-1} \sum_{j \neq i} DDI_A(x_i, x_j), \quad x_i, \quad x_j \in U.$$

3. Cautious ranking methodology

The process of cautious ranking methodology is shown in Fig. 1. In this section, we further elaborate each step of the systematic ranking approach.

3.1. Cautious dominance relation for interval data

Definition 3.1. Let $S=(U, AT, V, f)$ be an IOIS and $A \subseteq AT$. The *cautious dominance relation* with respect to A is defined as

$$R_A^{\trianglerighteq} = \{(y, x) \in U \times U \mid a_1^L(y) \geqslant a_1^L(x), \quad a_1^U(y) \geqslant a_1^U(x) (\forall a_1 \in A_1); \\ a_2^L(y) \leqslant a_2^L(x), \quad a_2^U(y) \leqslant a_2^U(x) (\forall a_2 \in A_2)\},$$

where $A_1 \cup A_2 = A$ (the attribute set A_1 denotes the profit type criteria and A_2 the cost type criteria).

Accordingly, the cautious dominance class $[x]_A^{\trianglerighteq}$, which denotes the set of objects dominating x with regard to cautious dominance relation $R_A^{\trianglerighteq}$, can be given as

$$[x]_A^{\trianglerighteq} = \{y \in U \mid a_1^L(y) \geqslant a_1^L(x), \quad a_1^U(y) \geqslant a_1^U(x) (\forall a_1 \in A_1); \\ a_2^L(y) \leqslant a_2^L(x), \quad a_2^U(y) \leqslant a_2^U(x) (\forall a_2 \in A_2)\} \\ = \{y \in U \mid (y, x) \in R_A^{\trianglerighteq}\}.$$

In general, the IOIS induced from $R_A^{\trianglerighteq}$ comprises two categories of forms, namely, the interval ordered decision table (IODT) and interval ordered information table (IOIT), defined as follows.

Definition 3.2. Let $S=(U, C \cup d, V, f)$ be an IOIS. If all the condition attributes of C are criteria, in which the domain of C denoted by $V_c (c \in C)$ is a set of interval numbers and the decision attribute d ($d \notin C, f(x, d) (x \in U)$ is single-valued) is an overall preference, then we call $S=(U, C \cup d, V, f)$ an IODT.

In an IODT, all the objects of U can be partitioned into s mutually exclusive classes by the decision attribute d, i.e., $D=\{D_1, D_2, \ldots, D_q, \ldots, D_r, \ldots, D_s\} (s \leqslant |U|)$, where it is assumed that the objects in D_r dominate the objects in D_q when $r > q (q, r \in G, G=\{1, 2, \ldots, s\})$.

Definition 3.3. Let $S=(U, C, V, f)$ be an IOIS. If all the condition attributes of C are criteria, in which the domain of C denoted by $V_c (c \in C)$ is a set of interval numbers, then we call $S=(U, C, V, f)$ an IOIT.

Remark 3.1. From its decision semantics, dominance relation $R_A^{\trianglerighteq}$ has a prudent characteristic, where we argue that object y is preferable to object x with regard to attribute set A if and only if the values of object y are more dominant than those of object x under all attributes belonging to A. Therefore, the dominance relation is used as the fundamental tool for describing the outranking relation between alternatives in the cautious ranking methodology.

3.2. Interval data-based data representation

In previous studies, decision analysis was frequently performed in terms of single-valued data. Nevertheless, the fluctuation of various attributes is a common phenomenon in real social and economic systems. In particular in the field of investment decisions, most indicators always show the characteristic of volatility. It is obvious that the typical representation of single-valued data cannot easily meet the requirement of describing the fluctuations of indicators in many practical environments. To address this problem, interval data-based data representation is expected to become a feasible method for reflecting the range of the indicators' values.

According to the idea of data-packaging, one can convert single-valued numbers into interval-valued numbers. By analyzing the range of the values of alternative x under attribute a based on its quarterly (or monthly or daily) data during a given period of time, in which the minimum value is represented by $a^L(x)$ and the maximum value by $a^U(x)$, we can obtain the interval data $[a^L(x), a^U(x)]$.

Remark 3.2. In this subsection, an interval data-based data representation method to reveal the entirety of the data set of attributes is presented. Clearly, the form of interval data has a greater capability to reflect the actual values of attributes than that of single-valued data. Here, a simple transformation is employed, because a limited amount of data samples is available in practical decision environments, especially for investment problems subjected to the current financial report standard. It can be imagined that we can adopt a superior estimation method for interval data by considering data distribution, if there is a sufficiently large amount of data samples for analysis with the improvement of the financial report standard.

3.3. Feature selection based on interval ordered conditional entropy

Considering the inherent nature of cautious ranking decisions, a measurement called interval ordered conditional entropy is presented to measure the ordered relevance between a condition attribute set and a decision attribute set.

From Definition 2.2, the formulation of interval ordered conditional entropy can be deduced as follows.

Definition 3.4. Let $S=(U, AT, V, f)$ be an IOIS and $A, B \subseteq AT$, $U/R_A^{\trianglerighteq} = \{[x_1]_A^{\trianglerighteq}, [x_2]_A^{\trianglerighteq}, \ldots, [x_{|U|}]_A^{\trianglerighteq}\}$, $U/R_B^{\trianglerighteq} = \{[x_1]_B^{\trianglerighteq}, [x_2]_B^{\trianglerighteq}, \ldots, [x_{|U|}]_B^{\trianglerighteq}\}$; then, the interval ordered conditional entropy of $A^{\trianglerighteq}$ with respect to $B^{\trianglerighteq}$ is defined as

$$E(A^{\trianglerighteq}|B^{\trianglerighteq}) = \sum_{i=1}^{|U|} \frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|} - \frac{|[x_i]_A^{\trianglerighteq} \cap [x_i]_B^{\trianglerighteq}|}{|U|}\right).$$

To clarify the nature of conducting feature selection based on interval ordered conditional entropy in an IODT, we further give several important definitions and theorems, as follows.

Definition 3.5. Let $S=(U, C \cup d, V, f)$ be an IODT and $D=\{D_1, D_2, \ldots, D_s\}$ $(s \leqslant |U|)$; then, the upward union and the downward union of class $D_q (1 \leqslant q \leqslant s)$ are defined as

$$D_q^{\trianglerighteq} = \bigcup_{r \geqslant q} D_r, \quad D_q^{\trianglelefteq} = \bigcup_{r \leqslant q} D_r.$$

Definition 3.6. Let $S=(U, C \cup d, V, f)$ be an IODT and $D=\{D_1, D_2, \ldots, D_s\}$ $(s \leqslant |U|)$; then, the dominance class of object x_i with respect to the decision attribute d is defined as

$$[x_i]_d^{\trianglerighteq} = \{x_j | x_j \in D_q^{\trianglerighteq} \quad (x_i \in D_q)\}.$$

Definition 3.7. Let $S=(U, C \cup d, V, f)$ be an IODT, $B \subseteq C$, $U/R_B^{\trianglerighteq} = \{[x_1]_B^{\trianglerighteq}, [x_2]_B^{\trianglerighteq}, \ldots, [x_{|U|}]_B^{\trianglerighteq}\}$, and $D=\{D_1, D_2, \ldots, D_s\}$ $(s \leqslant |U|)$; then, the interval ordered conditional entropy of $d^{\trianglerighteq}$ with respect to $B^{\trianglerighteq}$ is defined as

$$E(d^{\trianglerighteq}|B^{\trianglerighteq}) = \sum_{i=1}^{|U|} \frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|} - \frac{|[x_i]_B^{\trianglerighteq} \cap [x_i]_d^{\trianglerighteq}|}{|U|}\right).$$

Theorem 3.1 (Monotonicity). *Let $S=(U, C \cup d, V, f)$ be an IODT, $A, B \subseteq C$, $U/R_A^{\trianglerighteq} = \{[x_1]_A^{\trianglerighteq}, [x_2]_A^{\trianglerighteq}, \ldots, [x_{|U|}]_A^{\trianglerighteq}\}$, $U/R_B^{\trianglerighteq} = \{[x_1]_B^{\trianglerighteq}, [x_2]_B^{\trianglerighteq}, \ldots, [x_{|U|}]_B^{\trianglerighteq}\}$, and $D=\{D_1, D_2, \ldots, D_s\}$. If $|[x_i]_B^{\trianglerighteq} \cap [x_i]_d^{\trianglerighteq c}| \geqslant |[x_i]_A^{\trianglerighteq} \cap [x_i]_d^{\trianglerighteq c}|$ $(\forall x_i \in U)$, then $E(d^{\trianglerighteq}|B^{\trianglerighteq}) \geqslant E(d^{\trianglerighteq}|A^{\trianglerighteq})$.*

Proof. The proof is provided in Appendix A.

Remark 3.3. In the process described in this subsection, we aimed to select the pertinent condition attributes relevant to a decision attribute by characterizing the ordered relevance in an IODT. For $|[x_i]_B^{\trianglerighteq} \cap [x_i]_d^{\trianglerighteq c}|$, $[x_i]_B^{\trianglerighteq}$ represents the set of objects dominating object x_i with respect to the condition attribute set B and $[x_i]_d^{\trianglerighteq c}$ expresses the set of objects that are not as dominant as object x_i with regard to the decision attribute d. Therefore, it can be used to describe the ordered inconsistency between B and d. From Theorem 3.1, the larger is $|[x_i]_B^{\trianglerighteq} \cap [x_i]_d^{\trianglerighteq c}|$, the larger is $E(d^{\trianglerighteq}|B^{\trianglerighteq})$. This means that the larger is $E(d^{\trianglerighteq}|B^{\trianglerighteq})$, the greater is the inconsistency between B and d, and vice versa. Therefore, $E(d^{\trianglerighteq}|B^{\trianglerighteq})$ is a useful tool for describing the ordered relevance between the condition attribute set and the decision attribute set.

Theorem 3.2 (Minimum). *Let $S=(U, C \cup d, V, f)$ be an IODT, $B \subseteq C$, $U/R_B^{\trianglerighteq} = \{[x_1]_B^{\trianglerighteq}, [x_2]_B^{\trianglerighteq}, \ldots, [x_{|U|}]_B^{\trianglerighteq}\}$, and $D=\{D_1, D_2, \ldots, D_s\}$. Then, $[x_i]_B^{\trianglerighteq} \subseteq [x_i]_d^{\trianglerighteq}$ $(\forall x_i \in U)$ if and only if $E(d^{\trianglerighteq}|B^{\trianglerighteq})=0$.*

Proof. The proof is provided in Appendix B.

Theorem 3.3 (Maximum). *Let $S=(U, C \cup d, V, f)$ be an IODT, $B \subseteq C$, $U/R_B^{\trianglerighteq} = \{[x_1]_B^{\trianglerighteq}, [x_2]_B^{\trianglerighteq}, \ldots, [x_{|U|}]_B^{\trianglerighteq}\}$, and $D=\{D_1, D_2, \ldots, D_s\}$. Then, $[x_i]_B^{\trianglerighteq} \cap [x_i]_d^{\trianglerighteq} = x_i$ $(\forall x_i \in U)$ if and only if $E(d^{\trianglerighteq}|B^{\trianglerighteq}) = 1 - \frac{1}{|U|} - E(B^{\trianglerighteq})$.*

Proof. The proof is provided in Appendix C.

Remark 3.4. According to Theorem 3.2, when $[x_i]_B^{\trianglerighteq} \subseteq [x_i]_d^{\trianglerighteq}$ $(\forall x_i \in U)$, which represents that the condition attribute set B is completely consistent with the decision attribute d, $E(d^{\trianglerighteq}|B^{\trianglerighteq})$ reaches the minimum. According to Theorem 3.3, $E(d^{\trianglerighteq}|B^{\trianglerighteq})$ achieves the maximum when $[x_i]_B^{\trianglerighteq} \cap [x_i]_d^{\trianglerighteq} = x_i$ $(\forall x_i \in U)$, that is, the condition attribute set B is entirely inconsistent with the decision attribute d. From Theorems 3.1–3.3, we can draw the conclusion that the interval ordered conditional entropy is suitable for measuring the ordered relevance in IODTs.

On this basis, we further provide the definitions of attribute significance as follows.

Definition 3.8. Let $S=(U, C \cup d, V, f)$ be an IODT and $B \subseteq C$. The significance measure of $a (a \in B)$, called the inner significance, is defined by

$$Sig^{inner}(a, B, d) = E(d^{\trianglerighteq}|(B-a)^{\trianglerighteq}) - E(d^{\trianglerighteq}|B^{\trianglerighteq}).$$

Definition 3.9. Let $S=(U, C \cup d, V, f)$ be an IODT and $B \subseteq C$. The significance measure of $a (a \in C-B)$, called the outer significance, is defined by

$$Sig^{outer}(a, B, d) = E(d^{\trianglerighteq}|B^{\trianglerighteq}) - E(d^{\trianglerighteq}|(B \cup a)^{\trianglerighteq}).$$

Given an IODT, we say that attribute $a (a \in C)$ is indispensable if $Sig^{inner}(a, C, d) > 0$; otherwise, it is superfluous. The set of indispensable attributes is also called the core in rough set theory. On the basis of the core attributes, the key attribute subset, called the reduct, can be obtained through gradually adding selected attributes to the core. In the following, we provide the definition of the reduct based on the interval ordered conditional entropy.

Definition 3.10. Let $S=(U, C \cup d, V, f)$ be an IODT and $B \subseteq C$. We say that the attribute set B is a relative reduct if

(1) $E(d^{\trianglerighteq}|B^{\trianglerighteq}) = E(d^{\trianglerighteq}|C^{\trianglerighteq})$;
(2) $\forall a \in B$, $E(d^{\trianglerighteq}|(B-a)^{\trianglerighteq}) \neq E(d^{\trianglerighteq}|B^{\trianglerighteq})$.

Remark 3.5. From the first condition in Definition 3.10, it is found that the equal ordered relevance is preserved and the pertinent attributes with great significance can be extracted. Meanwhile, through the second condition in Definition 3.10, we can ensure the non-redundancy of the key attribute subset.

3.4. Two-step ranking mechanism

On the basis of the IOIT organized according to the key attribute subset achieved by using the feature selection method described above, in this subsection we present a two-step ranking mechanism for ranking alternatives.

On the basis of the cautious dominance relation $R_B^{\trianglerighteq}$, a cautious ranking result of alternatives can be obtained based on their comprehensive scores induced by the entire dominance degree $D_B(x_i)$. To meet the decision target of the cautious ranking decision, we set the comprehensive score based on $D_B(x_i)$ as the first ranking standard. However, some alternatives may be ranked in the same place in the above initial ranking order, and thus, this result will confuse decision makers. The following theorem further provides a proof of this shortcoming of the dominance degree.

Theorem 3.4. *Let $S=(U, C, V, f)$ be an IOIT and $B \subseteq C$, $x_i, x_j \in U$. If $(x_i, x_j) \in R_B^{\trianglerighteq}$, then $D_B(x_i, x_j)=1$.*

Proof. The proof is provided in Appendix D.

Remark 3.6. From Theorem 3.4, it is clearly found that the dominance degree $D_B(x_i, x_j)$ is identically equal to 1 when $(x_i, x_j) \in R_B^{\trianglerighteq}$. In other words, the preferability degree of object x_i over object x_j cannot be subtly measured if $x_i \succcurlyeq_B x_j$. Consequently, it is inevitable that some objects may locate in the same place in the ranking order.

In fact, the entire directional distance index $DDI_B(x_i)$ can produce a much finer ranking result, because it can more subtly characterize the distinction between objects from the numerical perspective. Moreover, it is also emphasized that the measurement of the weight of attribute a based on $E(a^{\trianglerighteq};B^{\trianglerighteq})$ in the definition of $DDI_B(x_i)$ has the characteristic of prudence because of its property of describing the ordered consistency between the rank induced by $R_a^{\trianglerighteq}$ and the cautious rank induced by $R_B^{\trianglerighteq}$. Therefore, the entire directional distance index is employed to achieve a more satisfactory outcome. Indeed, although the entire directional distance index can induce a considerably finer ranking result, we let $DDI_B(x_i)$ be the second ranking standard to ensure that the initial ranking order (i.e., the cautious ranking order) of the first step is unchanged. Hence, a two-step ranking mechanism can be constructed. The corresponding schematic diagram is illustrated in Fig. 2.

Based on the aforementioned analysis, the algorithm of the cautious ranking methodology is presented below.

Algorithm. Cautious ranking algorithm.

Input: An IODT $S=(U, C \cup d, V, f)$;
Output: A ranking order.

Step 1: $Red \longleftarrow \emptyset$ // Red is the pool to conserve the selected attributes;
Step 2: Compute $Sig^{inner}(a_k, C, d)$, $\forall a_k \in C$;
Step 3: Put a_k into Red, where $Sig^{inner}(a_k, C, d) > 0$ //These attributes constitute the core of the given decision table;
Step 4: While $E(d^{\trianglerighteq}|Red^{\trianglerighteq}) \neq E(d^{\trianglerighteq}|C^{\trianglerighteq})$, do
$\{Red \longleftarrow Red \cup \{a_0\}$, where $Sig^{outer}(a_0, Red, d) = max\{Sig^{outer}(a, Red, d), a \in C - Red\}\}$;
Step 5: Compute $Sig^{inner}(a', Red, d)$, $\forall a' \in Red$;
Step 6: $Red \longleftarrow Red - \{a'\}$, when $Sig^{inner}(a', Red, d)=0$;
Step 7: Set up an IOIT $S=(U, Red, V, f)$;
Step 8: Compute $D_{Red}(x_i)$ $(i=1, 2, \ldots, |U|)$ and determine the initial ranking order of the alternatives;
Step 9: Compute $DDI_{Red}(x_{l_j})$ $(j=1, 2, \ldots, h)$ if $D_{Red}(x_{l_1}) = D_{Red}(x_{l_2}) = \cdots = D_{Red}(x_{l_h})$, and determine the final ranking order of the alternatives.

During the era of the Great Depression in the 1930s, even conservative investors were subject to investment losses. On this formidable background, Graham and Dodd [49] proposed the Value Investment Theory for stock investment, and thereafter, both scholars and practitioners began to pay close attention to stock investment decisions [7,21,25,36–39,43,44].

On the basis of the Value Investment Theory, researchers found that the portfolios constructed with high ratios of book-to-market equity (B/M), earnings to price (E/P), or cash flow to price (C/P) have higher average returns than portfolios constructed with low B/M, E/P, or C/P [50,51]. However, some studies showed that value stocks are associated with the relative distress of firms [52], and the profitability of the value investment strategies appears nonstationary [53]. Thus, research on a cautious stock screening strategy is desirable, especially in the current increasingly competitive environment.

4.1. Decision process of cautious stock screening

According to the nature of cautious stock screening, we aimed to select the “good” stocks having a high level of coincidence between their financial performance and market success. Hence, an initial decision table, using financial indices as the condition attributes

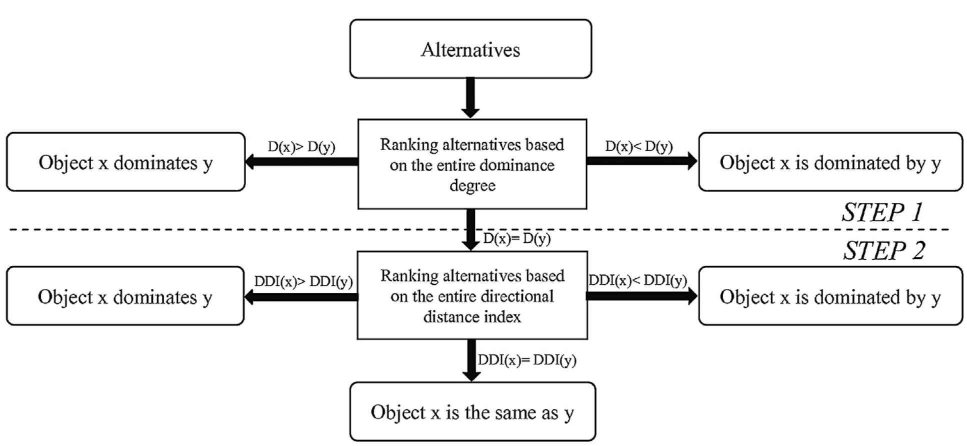


Fig. 2. Schematic diagram of two-step ranking mechanism.

and stock return as the decision attribute, is constructed to match the decision objective of the cautious stock selection program. Then, on the basis of interval data transformation for all attributes and a subsequent sorting process for the decision attribute, we can develop a new IODT, in which the values of the condition attributes are represented by interval data and the decision attribute appears in the form of ordered classes for extracting the key financial indices in the next phase.

Unlike in some past studies, where the "best" financial ratios were usually chosen from those applied in existing studies in the literature of stock selection, in our study we designed a feature selection method to select the representative financial ratios having significant ordered relevance to the stock return. Given the core idea of the cautious stock screening decision, the implementation of feature selection is desirable, because it meets the decision goal of screening out "good" stocks with the characteristic of a consistent excellent coincidence between their financial performance and market success.

In terms of the above selected key financial ratios, an IOIT can be organized. Through the two-step ranking mechanism provided based on the cautious dominance relation for interval data, we can obtain the ranking results of cautious stock screening.

To clearly illustrate the decision process, a realistic case, involving the 180 firms, the stocks of which are components of the Shanghai Stock Exchange 180 Index (SSE 180) of China, in the period from 2009 to 2011, that is, after the outbreak of the global financial crisis, was employed in this study. The reason for selecting this period is that the market suffered from economic fluctuations during that time [21] and the corresponding results achieved in this situation are more trustworthy. On the background of the financial crisis, Sánchez-Monedero et al. presented a pairwise class distances (PCD) projection and a PCD-based classifier in which standard regression techniques were applied to handle the sovereign credit rating classification problem and validated the quality of the proposed projection by using sovereign rating data of 27 countries in the European Union during the period 2007–2010 [54]. Considering the decrease in the interval between great market crises (e.g., between the financial crisis of 1997–1998 and the dot.com crisis of 2000), Sevastjanov and Dymova proposed a new stock ranking method and chose a relatively short investment horizon of 1 year. In particular taking into account that the aim of the study was to present only a new approach for stock screening, the authors chose sample data covering the period of two adjacent years [37]. Therefore, the experimental data used in our study covered only the period from 2008 to 2011. Meanwhile, since the design goal of the SSE 180 index is to establish a benchmark of investment performance evaluation, a case study based on this index was considered suitable for the purpose of this study.

The experimental procedure was divided into two stages: screening and testing. To screen out good stocks, the firms' quarterly financial data in the previous year and monthly stock return in the following year were employed. Since the deadline of the annual financial statement publication of public firms is April 30 of the subsequent year, the monthly stock return of May and June of year t was employed for matching the quarterly financial data of year $t-1$. After the screening stage, a testing stage, in which the monthly data of stock return during the period from July to December of year t were employed, was conducted to determine whether the returns of the selected "good" stocks could outperform those of the benchmark or not. In this case study, the return in the SSE 180, which represents the average market return, was used as the benchmark of the investment performance evaluation, because of its additional advantage of alleviating the impact of different macromarket conditions. The time distribution of the three successive rounds of tests in this case study is shown in Fig. 3.

4.2. Initial decision table considering financial performance and market return

According to the intrinsic mechanism of the cautious stock screening, an initial decision table was constructed by simultaneously considering financial performance and market return. In this decision table, the relevant financial ratios constituted the condition attribute set and the indicator of the market return was used as the decision attribute.

As indicators of the financial performance, some common ratios from the well known textbooks such as [55] and other literature [25,39] were adopted. It should be noted that we did not deliberately choose all the best ratios having significant relevance to the market return of stocks demonstrated in existing studies in the literature. We argue that the selection of common ratios was more propitious for verifying the validity of the proposed approach. In this study, the chosen financial ratios comprised 11 common indicators that respectively reflect 5 aspects of financial performance, i.e., earning capacity, the ability to make cash flow, operating capacity, growth capability, and solvency. Table 1 lists a detailed description of these financial ratios $c_l (l = 1, 2, \ldots, 11)$. For decision attribute d, the stock return was used as the measurement of the market return of stocks denoted by $MR_{i,t}$, which equals $P_{i,t}/P_{i,t-1} - 1$, where $P_{i,t}$ represents the comparable closing price of stock x_i on the last trading day of the tth period.

4.3. Interval data transformation and sorting process

To reflect the ranges of the values of the related indicators more comprehensively, single-valued data in the initial decision table were transformed into interval-valued data. Because of the financial report standard of China, we can obtain only quarterly data of financial ratios from the quarterly financial statement of a firm (or stock) x_i. Through the method of interval data transformation (see Section 3.2), the value of c_l of stock x_i takes on the interval-valued scale. For the decision attribute, we applied the same transformation method to monthly data of stock returns. The interval data table of the first test round of the case is shown in Table 2, in which the stocks are components of the SSE 180 in year t (2009), the interval data of financial performance were induced from the quarterly data of year $t-1$ (2008), and the stock returns represented by the form of interval data were derived from the single-valued data of

Rounds	Screening Stage						Testing Stage					
	Financial Indicators				Stock Return		Accumulated Return					
1st	Year 2008				Year 2009		Year 2009					
	Q1	Q2	Q3	Q4	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2nd	Year 2009				Year 2010		Year 2010					
	Q1	Q2	Q3	Q4	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
3rd	Year 2010				Year 2011		Year 2011					
	Q1	Q2	Q3	Q4	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec

Fig. 3. Time distribution of the three successive rounds of tests.

Table 1
Indicators of financial performance.

Financial performance	Code	Ratios	Description
Earning Capacity	c_1	Operating Margin (OM)	Operating profits/Operating income
	c_2	Net Profit Margin of Total Assets (NPMTA)	Net profit/Average total assets
	c_3	Return on Equity (ROE)	Net profit/Average shareholders' equity
Cash Flow	c_4	Operating Activities (OA)	Net cash flow from operating activities/Current liabilities
	c_5	Cash Ratio (CR)	Monetary assets/Current liabilities
Operating Capacity	c_6	Total Assets Turnover (TAT)	Operating income/Average total assets
	c_7	Fixed Asset Turnover (FAT)	Operating income/Average fixed assets
Growth Capability	c_8	Growth Rate of Total Assets (GRTA)	(Ending total assets-Beginning total assets)/Beginning total assets
	c_9	Growth Rate of Net Profit (GRNT)	(Net profit in current period-Net profit in previous period)/Net profit in previous period
Solvency	c_{10}	Quick Ratio (QR)	(Current assets-Inventories)/Current liabilities
	c_{11}	Ratio of Liabilities to Assets (RLA)	Total liabilities/Total assets

Table 2
Interval data table of the first test round of the case study.

U	c_1	c_2	...	c_{11}	d
x_1	[0.009152, 0.015839]	[0.017419, 0.029473]	...	[0.519397, 0.544313]	[0.046750, 0.343487]
x_2	[0.009635, 0.019480]	[0.013793, 0.028261]	...	[0.658888, 0.712090]	[0.010204, 0.119533]
x_3	[0.016156, 0.031490]	[0.026645, 0.052360]	...	[0.596715, 0.608825]	[−0.061224, − 0.008343]
x_4	[0.021943, 0.067430]	[0.030614, 0.093865]	...	[0.701279, 0.748481]	[0.053089, 0.053472]
x_5	[0.013569, 0.044635]	[0.031694, 0.106636]	...	[0.413960, 0.442810]	[−0.028714, 0.108161]
x_6	[0.019411, 0.042056]	[0.036965, 0.068780]	...	[0.506051, 0.537983]	[0.036991, 0.113436]
x_7	[0.011146, 0.022827]	[0.014252, 0.029973]	...	[0.741459, 0.774778]	[0.103852, 0.112689]
x_8	[0.006434, 0.024131]	[0.022877, 0.084960]	...	[0.266577, 0.297923]	[0.100927, 0.436148]
x_9	[0.014325, 0.109014]	[0.021999, 0.174011]	...	[0.602231, 0.685546]	[−0.075556, 0.016296]
x_{10}	[0.000374, 0.008701]	[0.000997, 0.020861]	...	[0.333118, 0.447271]	[0.058096, 0.227962]
...	...	...	...	...	...

Table 3
Interval ordered decision table of the first test round of the case study.

U	c_1	c_2	...	c_{11}	d
x_1	[0.009152, 0.015839]	[0.017419, 0.029473]	...	[0.519397, 0.544313]	3
x_2	[0.009635, 0.019480]	[0.013793, 0.028261]	...	[0.658888, 0.712090]	2
x_3	[0.016156, 0.031490]	[0.026645, 0.052360]	...	[0.596715, 0.608825]	1
x_4	[0.021943, 0.067430]	[0.030614, 0.093865]	...	[0.701279, 0.748481]	2
x_5	[0.013569, 0.044635]	[0.031694, 0.106636]	...	[0.413960, 0.442810]	2
x_6	[0.019411, 0.042056]	[0.036965, 0.068780]	...	[0.506051, 0.537983]	2
x_7	[0.011146, 0.022827]	[0.014252, 0.029973]	...	[0.741459, 0.774778]	3
x_8	[0.006434, 0.024131]	[0.022877, 0.084960]	...	[0.266577, 0.297923]	3
x_9	[0.014325, 0.109014]	[0.021999, 0.174011]	...	[0.602231, 0.685546]	1
x_{10}	[0.000374, 0.008701]	[0.000997, 0.020861]	...	[0.333118, 0.447271]	2
...	...	...	...	...	...

May and June of year t (2009). In addition, for a more reasonable screening decision, three types of samples needed to be deleted. (1) The samples in the financial sector were excluded because their capital structures are distinctly different from those of other samples in the nonfinancial sector; (2) the samples with negative net profit were eliminated from the sample set for the sake of a cautious investment target; and (3) the samples with missing values were also deleted in this case study. Accordingly, the sample sizes in the three successive rounds of tests were 91, 116, and 128, respectively.

Indeed, the cautious stock screening issue can be viewed as a selection process that relies on financial ratios and stock return in the context of ordered semantics. From this viewpoint, we need to sort the values of the decision attribute (stock return) of alternative stocks. Without loss of generality, alternatives can be sorted into three ordered classes: D_1, D_2, and D_3, where D_1 comprises the stocks located in the bottom 30%, D_3 the stocks situated in the top 30%, and D_2 the remaining stocks. Here, the two-step ranking mechanism was performed to sort the interval-valued data of the stock return. For convenience, the values of D_1, D_2, and D_3 were conventionally set to be 1, 2, and 3, respectively, and then Table 2 was converted into Table 3, that is, an IODT for stock screening. Table 4 presents the descriptive statistics for the related data samples.

4.4. Extracting the key financial indicators and screening out "good" stocks

Continuing from the IODT shown in Table 3 and using Steps 1–6 of the proposed algorithm, we could obtain three key attribute sets, the so-called reducts, having the same conditional entropy and consisting of the seven financial indices shown in Table 5. By analyzing these results, we achieved three key attribute sets cover-

Table 4
Descriptive statistics for data samples in the first test round of the case study.

	Min	*Max*	*Mean*	*Std. Dev.*
c_1	0.000073	0.158075	0.022560	0.017534
c_2	0.000168	0.311692	0.045325	0.033171
c_3	0.000882	1.299880	0.222929	0.176644
c_4	−0.405221	1.058395	0.089707	0.149284
c_5	0.041321	14.711670	0.669713	0.970417
c_6	0.005748	2.108561	0.171091	0.174165
c_7	0.025880	101.166385	3.469648	8.401274
c_8	−0.285174	1.695945	0.311459	0.392318
c_9	−0.552297	16.785282	0.571784	1.932438
c_{10}	0.065356	15.603957	1.090511	1.071685
c_{11}	0.165540	0.943306	0.517180	0.160230

Table 5
Three key attribute sets of the first test round of the case study.

1	2	3
NPMTA	*NPMTA*	*NPMTA*
OA	*OA*	*OA*
FAT	*FAT*	*FAT*
GRTA	*GRTA*	*GRTA*
GRNT	*GRNT*	*GRNT*
RLA	*RLA*	*RLA*
ROE	*CR*	*QR*

Table 6
Stock codes of the top 10 selected stocks of the 3 ranking results in the first test round of the case study.

1	2	3	1 *and* 2 *and* 3
600519	600519	600519	600519
601958	601958	601958	601958
601898	601898	601898	601898
600007	600007	600026	600309
600309	600026	600547	600547
600547	600309	600309	601857
601857	600508	600508	600508
600508	600547	601001	601001
601001	601001	601857	
601699	601857	600489	

ing all five categories of financial performance ratios. It is obvious that the selected pertinent indices can comprehensively reflect the firms' financial situation. Furthermore, we can also clearly see that only one indicator in the three feature selection results is different from one another, as shown in the last row of Table 5. In brief, using the above feature selection method it is possible to effectively explore the representative financial indicators that have significant ordered relevance to the stock return.

Furthermore, by using the two-step ranking mechanism mentioned above, cautious stocks ranking results can be obtained from the IOIT organized according to the key financial indicator set. Continuing the above case study, we list in Table 6 the top 10 stocks of the year 2009, i.e., the first round of the test, based on the three different key attribute sets. It is worth noting that the same 8 stocks appear in the top 10 stocks among the 3 ranking results, which, to some extent, demonstrates the robustness of the proposed cautious ranking approach.

To verify the validity of the proposed method, we further calculated the accumulated return AMR_i, denoted by $AMR_i = \prod_{t=1}^{T}(1 + MR_{i,t}) - 1$ where T represents the holding periods of stock i, of the top 10 stocks during the period from July (holding 1 month) to December (holding 6 months) of the year 2009. Meanwhile, the return of SSE180 over the same period is introduced as the benchmark with which to compare the investment performance.

Tables 7–9 illustrate the profitability of the presented systematic approach in terms of the three cautious ranking results (abbreviated as C-ranking) for the year 2009. To show a clearer comparison of their investment performance, the corresponding results are represented in the graphs in Fig. 4(a)–(c). It is clear that the selected stocks, of which only a few (e.g., stocks 601857 and 600489) produce a negative return, can achieve a higher average return than the SSE180. Furthermore, when considering only these eight stocks jointly contained in the three ranking results in Table 6, we can achieve the best investment performance from a 1-month to a 6-month holding period, namely, 0.214070, −0.066907, 0.022719, 0.133713, 0.263313, and 0.237335, respectively. Moreover, this cautious screening procedure was conducted also in the second and third rounds of tests. In each of these two rounds, there was only one reduct and a corresponding ranking result. Accordingly, their results are shown in the graphs in Fig. 4(d) and (e), which also provide evidence supporting the effectiveness of the developed methodological approach. In brief, it is apparent that the proposed ranking methodology can outperform the benchmark and provide a credible investment decision-making method for cautious investors.

For further performance comparison, the three most popular indicators, namely, B/M, E/P, C/P, which have been documented as

Table 7
Top 10 stocks returns obtained by the first ranking result in the first test round of the case study.

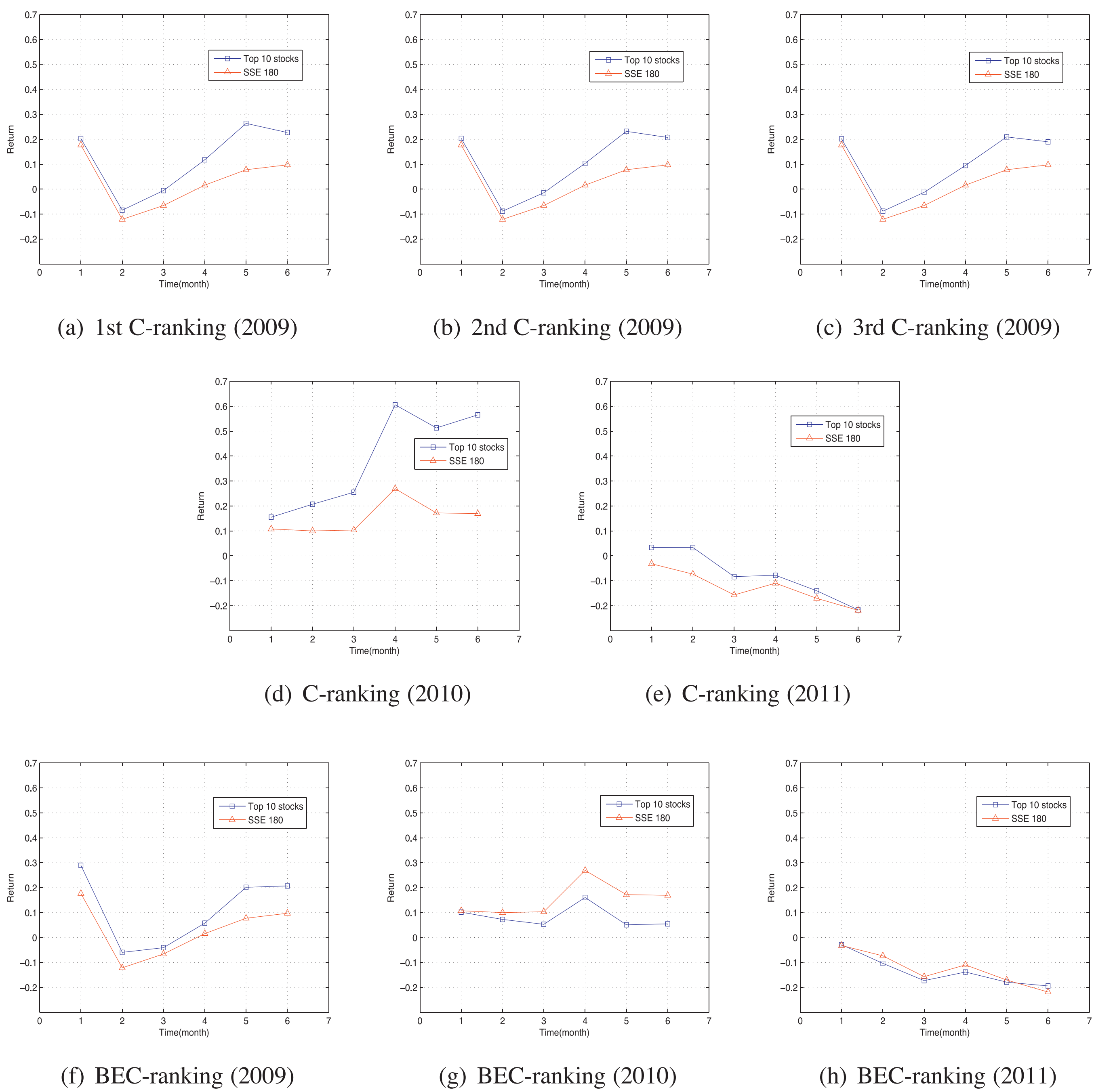
Stock code	1	2	3	4	5	6
600519	0.021624	0.021420	0.122683	0.073311	0.184313	0.156460
601958	0.385139	0.042724	0.158514	0.274922	0.321361	0.180184
601898	0.298250	−0.090241	−0.065653	0.065484	0.143346	0.113021
600007	−0.018966	−0.239655	−0.193104	−0.061207	0.133620	0.058620
600309	0.115409	−0.062778	0.093215	0.257450	0.480659	0.522512
600547	0.015141	−0.201997	−0.018303	0.089518	0.367888	0.336108
601857	0.086326	−0.116022	−0.115011	−0.072504	−0.061355	−0.036965
600508	0.426689	0.011532	0.068644	0.208677	0.381659	0.381110
601001	0.363981	−0.139894	−0.062332	0.172846	0.288635	0.246253
601699	0.332742	−0.071157	−0.048874	0.164851	0.392757	0.311218
Average stocks returns	0.202634	−0.084607	−0.006022	0.117335	0.263288	0.226852
Returns of SSE 180	0.176642	−0.121437	−0.066014	0.015933	0.077496	0.097463

Table 8
Top 10 stocks returns obtained by the second ranking result in the first test round of the case study.

Stock code	1	2	3	4	5	6
600519	0.021624	0.021420	0.122683	0.073311	0.184313	0.156460
601958	0.385139	0.042724	0.158514	0.274922	0.321361	0.180184
601898	0.298250	−0.090241	−0.065653	0.065484	0.143346	0.113021
600007	−0.018966	−0.239655	−0.193104	−0.061207	0.133620	0.058620
600026	0.341596	−0.110766	−0.134003	0.021690	0.076686	0.111543
600309	0.115409	−0.062778	0.093215	0.257450	0.480659	0.522512
600508	0.426689	0.011532	0.068644	0.208677	0.381659	0.381110
600547	0.015141	−0.201997	−0.018303	0.089518	0.367888	0.336108
601001	0.363981	−0.139894	−0.062332	0.172846	0.288635	0.246253
601857	0.086326	−0.116022	−0.115011	−0.072504	−0.061355	−0.036965
Average stocks returns	0.203519	−0.088568	−0.014535	0.103019	0.231681	0.206885
Returns of SSE 180	0.176642	−0.121437	−0.066014	0.015933	0.077496	0.097463

Table 9
Top 10 stocks returns obtained by the third ranking result in the first test round of the case study.

Stock code	1	2	3	4	5	6
600519	0.021624	0.021420	0.122683	0.073311	0.184313	0.156460
601958	0.385139	0.042724	0.158514	0.274922	0.321361	0.180184
601898	0.298250	−0.090241	−0.065653	0.065484	0.143346	0.113021
600026	0.341596	−0.110766	−0.134003	0.021690	0.076686	0.111543
600547	0.015141	−0.201997	−0.018303	0.089518	0.367888	0.336108
600309	0.115409	−0.062778	0.093215	0.257450	0.480659	0.522512
600508	0.426689	0.011532	0.068644	0.208677	0.381659	0.381110
601001	0.363981	−0.139894	−0.062332	0.172846	0.288635	0.246253
601857	0.086326	−0.116022	−0.115011	−0.072504	−0.061355	−0.036965
600489	−0.038251	−0.240589	−0.176533	−0.145720	−0.091985	−0.115361
Average stocks returns	0.201590	−0.088661	−0.012878	0.094567	0.209121	0.189486
Returns of SSE 180	0.176642	−0.121437	−0.066014	0.015933	0.077496	0.097463



(a) 1st C-ranking (2009)
(b) 2nd C-ranking (2009)
(c) 3rd C-ranking (2009)
(d) C-ranking (2010)
(e) C-ranking (2011)
(f) BEC-ranking (2009)
(g) BEC-ranking (2010)
(h) BEC-ranking (2011)

Fig. 4. Returns for comparison.

Table 10
Key attribute sets of the three test rounds based on the feature selection method in [3].

1	2	3
NPMTA	*NPMTA*	*NPMTA*
ROE	*ROE*	*ROE*
OA	*OA*	*OA*
CR	*CR*	*CR*
TAT	*TAT*	*TAT*
FAT	*FAT*	*FAT*
GRTA	*GRTA*	*GRTA*
GRNT	*GRNT*	*GRNT*
QR	*QR*	*QR*
RLA	*RLA*	*RLA*

having significant relevance to stock return [50,51], were employed to organize an IOIT. By using the two-step ranking mechanism, we could achieve the corresponding ranking results (abbreviated as BEC-ranking). The average returns of the top 10 stocks over the same period based on the same data samples were calculated; the results of the three rounds of tests are shown in the graphs in Fig. 4(f)–(h). Although the top 10 stocks in the first test round of the case study obtained by the 3 most popular indices produce higher returns than the benchmark, in the second succeeding test round the performance was worse and the selected stocks of the last round of the test could achieve only approximate returns as compared with the benchmark. Further analysis of all these results shows clearly that the process of feature selection is quite necessary for cautious stock screening, and the presented systemic approach in view of a cautious ranking decision has the capability to achieve a higher average return.

4.5. Further discussion

According to the related investment theories, the risk of the investment needs to be considered when conducting a performance comparison. The Sharpe ratio is a classical risk-return measurement proposed in [56], which measures the average excess return on a risk-free return per unit of standard deviation in a portfolio. Consequently, the Sharpe ratio was adopted for comparing the investment profitability of different portfolios more reasonably.

Meanwhile, to analyze in depth the effect of the methodology presented in this study, three additional significant procedures were introduced as follows. First, to validate its effectiveness the proposed feature selection algorithm was experimentally compared with a second feature selection method. Here, we introduced an attribute reduction approach based on the discernibility matrix presented in [3] to make a further comparison. Second, the results were compared with those produced when no feature selection algorithm was applied. Third, a new benchmark was employed with which to further compare the experimental results. In the previous subsection, we described the application of the benchmark based on SSE180 in the performance comparison (see Section 4.4). However, since some samples were deleted to achieve more reasonable screening decisions, a new benchmark based on an underlying subsample of SSE180 (i.e., 91, 116, and 128 stocks, respectively) was used to conduct supplementary comparisons.

Table 10 shows that Qian's method in [3] produces the same feature selection results in the three test rounds and has more features (10) than the algorithm proposed in this study (i.e., 7, 9, and 10 features, respectively). Naturally, based on the above key attribute sets, we can obtain the corresponding ranking results by using also the two-step ranking mechanism (abbreviated as Qian-ranking). In addition, when no feature selection algorithm was applied, that is, all indicators are directly used to rank alternatives, the corresponding results can also be obtained (abbreviated as NoFS-ranking). Table 11 shows the Sharpe ratios of the top 10 stocks in the 4 classes of ranking results in comparison with those of the two benchmarks. The results of the Sharpe ratios clearly show that C-ranking is superior to other models and can outperform the two benchmarks (highlighted in bold in Table 11), except for the case of the year 2011. In 2011, the excess returns and standard deviations of the four models and two benchmarks are nearly equal. Further analysis of the results for this year shows that the excess return and standard deviation of C-ranking is the same as that of the benchmark of SSE180 and its standard deviation is equal to that of the benchmark of the subsample of SSE180, but its excess return is slightly higher.

Table 11
Sharpe ratios for performance comparison.

	2009	2009	2009	2010	2011
C-ranking	**0.33**	**0.31**	**0.29**	**0.79**	−0.04/0.05
BEC-ranking	0.27	0.27	0.27	0.13	−0.04/0.04
Qian-ranking	0.27	0.27	0.27	**0.79**	−0.04/0.05
NoFS-ranking	0.27	0.27	0.27	**0.79**	−0.04/0.05
*Subsample of SSE*180	0.25	0.25	0.25	0.59	−0.05/0.05
*SSE*180	0.18	0.18	0.18	0.36	−0.04/0.05

In addition, the necessity of utilizing the two-step ranking mechanism is now discussed. Indeed, the first ranking standard of the two-step mechanism was introduced in [3,57]. However, it does not provide a complete rank. In this paper, Theorem 3.1 proved this shortcoming of the entire dominance degree. To address this problem, we constructed a two-step ranking mechanism. In fact, some objects are frequently located in the same place in the ranking results induced by the entire dominance degree. In all the four classes of ranking results, there are always some stocks lying in the same place. For example, in the C-ranking result of 2010, there are 2 stocks tied for 5th place, 2 for 7th place, and 10 for 9th place. The values of $D_B(x_i)$ in this example are listed as follows.

$$\begin{aligned} D_B(x_{14}) &= D_B(x_{100}) = 0.991604 \quad (\textit{tied for 5th place}), \\ D_B(x_{43}) &= D_B(x_{78}) = 0.991528 \quad (\textit{tied for 7th place}), \\ D_B(x_2) &= D_B(x_{15}) = D_B(x_{35}) = D_B(x_{53}) = D_B(x_{54}) = D_B(x_{85}) = D_B(x_{102}) = D_B(x_{112}) \\ &= D_B(x_{113}) = D_B(x_{114}) = 0.0.991454 \quad (\textit{tied for 9th place}). \end{aligned}$$

By applying the two-step ranking mechanism, we can easily obtain a complete ranking order. Obviously, it is useful to select the top 10 stocks in every constructed portfolio. More generally, a complete ranking result helps decision makers create a more satisfactory decision scheme.

In summary, it is important and useful to develop a systematic ranking methodology. Feature selection is an important preprocessing step for selecting informative features related to decision targets. As a type of typical feature selection approach, the proposed attribute reduction algorithm in the framework of rough set theory can select fewer features and perform better than the comparative approach. In fact, certain decision problems usually have specific decision targets. The corresponding feature selection algorithm related to the specific decision targets is useful for discovering informative features, the relevance of which to the decision objectives is significant. Correspondingly, using the feature selection algorithm, we can not only remove redundant features but also improve the decision performance. The performance comparison of C-ranking and NoFS-ranking (see Table 11) further supports the above claims.

Based on the selected key attributes, a cautious two-step ranking mechanism is used to construct the portfolios. In the existing studies in the literature mentioned above [22,23], the researchers usually focused on specific views to depict the risk aversion attitude of decision makers in the decision-making process. In [22,58], only the weights of decision makers from the perspective of risk aversion for group decision making, which can reflect the common decision aspirations and consistent judgement of decision makers, were considered. In [23,59], the risk aversion utility function was introduced to depict the decision makers' risk attitude, but the parameter estimation remains a challenging issue. Unlike in these studies, in this study the proposed ranking mechanism introduced risk aversion behaviors into each step of the ranking process, such as cautious dominance relation and cautious attribute weights, and the initial ranking order (i.e., cautious ranking order) was kept unchanged. Moreover, based on the cautious dominance relation, the cautious attribute weights and the cautious ranking order could be directly obtained from the data. In summary, in this section, the conjoint analysis involving key feature subsets, ranking order, and performance comparison validated the effectiveness of the cautious ranking methodology proposed in this paper.

5. Conclusions

Considering the preferences of decision makers and special decision tasks, in this study a novel ranking methodology was constructed. The highlights and contributions of this study are summarized as follows. (1) A cautious ranking methodological approach that considers risk aversion behavior in each step of problem solving was built. Furthermore, it provides a new paradigm of problem solving for risk avoiders. (2) An ordered-relevance-preserving feature selection approach was built to select informative indicators and it was shown to achieve a better performance. By using the proposed feature selection framework, one can effectively tackle the problems with special decision objectives. (3) A ranking methodology taking into account decision behavior, data representation, feature selection, and the ranking mechanism was established. More generally, the integrated mechanism offers a new research perspective for decision modeling. (4) The case study reported in this paper is the first attempt, according to our uncomprehensive search, to apply interval data to screen out "good" stocks. We applied the proposed cautious ranking methodology to stock screening decisions through three rounds of tests. The experimental results indicate that the proposed methodological approach is effective.

However, our study also has the corresponding limitation. The cautious dominance relation proposed in this paper provides only a fixed representation of a risk aversion attitude. In general, different decision makers may have different degrees of risk preference. However, it should be favorably acknowledged that the cautious dominance relation in this study affords a basic representation paradigm for risk avoiders.

Future research directions include, but are not limited to: (1) building the adjustable dominance relation considering different degrees of risk preference, and (2) developing the corresponding risk attitudinal ranking methodology with hybrid data forms.

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Appendix A

Theorem 3.1 (Monotonicity). *Let* $S=(U, C\cup d, V, f)$ *be an IODT,* $A, B\subseteq C$, $U/R_A^{\trianglerighteq}=\{[x_1]_A^{\trianglerighteq}, [x_2]_A^{\trianglerighteq}, \ldots, [x_{|U|}]_A^{\trianglerighteq}\}$, $U/R_B^{\trianglerighteq}=\{[x_1]_B^{\trianglerighteq}, [x_2]_B^{\trianglerighteq}, \ldots, [x_{|U|}]_B^{\trianglerighteq}\}$, *and* $D=\{D_1, D_2, \ldots, D_s\}$. *If* $|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq c}|\geqslant|[x_i]_A^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq c}|$ $(\forall x_i\in U)$*; then,* $E(d^{\trianglerighteq}|B^{\trianglerighteq})\geqslant E(d^{\trianglerighteq}|A^{\trianglerighteq})$ [34].

Proof. Suppose that $|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq c}|\geqslant|[x_i]_A^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq c}|$ $(\forall x_i\in U)$, then, we have that

$$\begin{aligned}
E(d^{\trianglerighteq}|B^{\trianglerighteq}) &= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\right)\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\frac{|[x_i]_B^{\trianglerighteq}-[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\frac{|[x_i]_B^{\trianglerighteq}\cap U-[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\frac{|[x_i]_B^{\trianglerighteq}\cap(U-[x_i]_d^{\trianglerighteq})|}{|U|} = \sum_{i=1}^{|U|}\frac{1}{|U|}\frac{|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq c}|}{|U|}\\
&\geqslant \sum_{i=1}^{|U|}\frac{1}{|U|}\frac{|[x_i]_A^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq c}|}{|U|} = \sum_{i=1}^{|U|}\frac{1}{|U|}\frac{|[x_i]_A^{\trianglerighteq}\cap(U-[x_i]_d^{\trianglerighteq})|}{|U|}\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\frac{|[x_i]_A^{\trianglerighteq}-[x_i]_A^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_A^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_A^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\right) = E(d^{\trianglerighteq}|A^{\trianglerighteq})
\end{aligned}$$

This completes the proof.

Appendix B

Theorem 3.2 (Minimum). *Let* $S=(U, C\cup d, V, f)$ *be an IODT,* $B\subseteq C$, $U/R_B^{\trianglerighteq}=\{[x_1]_B^{\trianglerighteq}, [x_2]_B^{\trianglerighteq}, \ldots, [x_{|U|}]_B^{\trianglerighteq}\}$, *and* $D=\{D_1, D_2, \ldots, D_s\}$. *Then,* $[x_i]_B^{\trianglerighteq}\subseteq[x_i]_d^{\trianglerighteq}$ $(\forall x_i\in U)$ *if and only if* $E(d^{\trianglerighteq}|B^{\trianglerighteq})=0$.

Proof. (1) "⇒" Suppose that $[x_i]_B^{\trianglerighteq}\subseteq[x_i]_d^{\trianglerighteq}$ $(\forall x_i\in U)$; then, we have that

$$\begin{aligned}
E(d^{\trianglerighteq}|B^{\trianglerighteq}) &= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\right)\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}\right)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
E(d^{\trianglerighteq}|B^{\trianglerighteq}) &= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\right)\\
&= \sum_{i=1,i\neq k_h}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\right)\\
&\quad+\sum_{h=1}^{m}\frac{1}{|U|}\left(\frac{|[x_{k_h}]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_{k_h}]_B^{\trianglerighteq}\cap[x_{k_h}]_d^{\trianglerighteq}|}{|U|}\right)\\
&= \sum_{i=1,i\neq k_h}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}|}{|U|})+\sum_{h=1}^{m}\frac{1}{|U|}(\frac{|[x_{k_h}]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_{k_h}]_B^{\trianglerighteq}\cap[x_{k_h}]_d^{\trianglerighteq}|}{|U|}\right)\\
&> \sum_{i=1,i\neq k_h}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}|}{|U|})+\sum_{h=1}^{m}\frac{1}{|U|}(\frac{|[x_{k_h}]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_{k_h}]_B^{\trianglerighteq}|}{|U|}\right)\\
&= 0
\end{aligned}$$

Appendix C

Theorem 3.3 (Maximum). *Let* $S=(U, C\cup d, V, f)$ *be an IODT,* $B\subseteq C$, $U/R_B^{\trianglerighteq}=\{[x_1]_B^{\trianglerighteq}, [x_2]_B^{\trianglerighteq}, \ldots, [x_{|U|}]_B^{\trianglerighteq}\}$, *and* $D=\{D_1, D_2, \ldots, D_s\}$. *Then,* $[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}=x_i \quad (\forall x_i\in U)$ *if and only if* $E(d^{\trianglerighteq}|B^{\trianglerighteq})=1-\frac{1}{|U|}-E(B^{\trianglerighteq})$.

Proof. (1) "⇒" Suppose that $[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}=x_i \quad (\forall x_i\in U)$; then, we have that

$$\begin{aligned}
E(d^{\trianglerighteq}|B^{\trianglerighteq}) &= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\right)\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|x_i|}{|U|}\right)=\sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{1}{|U|}\right)\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-1+1-\frac{1}{|U|}\right)\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{1}{|U|}\right)-\sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}\right)\\
&= 1-\frac{1}{|U|}-E(B^{\trianglerighteq})
\end{aligned}$$

$$\begin{aligned}
E(d^{\trianglerighteq}|B^{\trianglerighteq}) &= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\right)\\
&= \sum_{i=1, i\neq k_h}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_i]_B^{\trianglerighteq}\cap[x_i]_d^{\trianglerighteq}|}{|U|}\right)\\
&\quad+\sum_{h=1}^{m}\frac{1}{|U|}\left(\frac{|[x_{k_h}]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_{k_h}]_B^{\trianglerighteq}\cap[x_{k_h}]_d^{\trianglerighteq}|}{|U|}\right)\\
&= \sum_{i=1, i\neq k_h}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|x_i|}{|U|}\right)\\
&\quad+\sum_{h=1}^{m}\frac{1}{|U|}\left(\frac{|[x_{k_h}]_B^{\trianglerighteq}|}{|U|}-\frac{|[x_{k_h}]_B^{\trianglerighteq}\cap[x_{k_h}]_d^{\trianglerighteq}|}{|U|}\right)\\
&< \sum_{i=1, i\neq k_h}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{|x_i|}{|U|}\right)\\
&\quad+\sum_{h=1}^{m}\frac{1}{|U|}\left(\frac{|[x_{k_h}]_B^{\trianglerighteq}|}{|U|}-\frac{|x_{k_h}|}{|U|}\right)\\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(\frac{|[x_i]_B^{\trianglerighteq}|}{|U|}-\frac{1}{|U|}\right)=1-\frac{1}{|U|}-E(B^{\trianglerighteq}
\end{aligned}$$

Appendix D

Theorem 3.4. *Let* $S=(U, C, V, f)$ *be an IOIT and* $B\subseteq C$, $x_i, x_j\in U$. *If* $(x_i, x_j)\in R_B^{\trianglerighteq}$, *then* $D_B(x_i, x_j)=1$.

Proof. If $(x_i, x_j)\in R_B^{\trianglerighteq}$, it follows from the cautious dominance class $[x_j]_B^{\trianglerighteq}$ that $x_i\in[x_j]_B^{\trianglerighteq}$. Then, we have that $[x_i]_B^{\trianglerighteq}\subseteq[x_j]_B^{\trianglerighteq}$.

Thus,

$$\begin{aligned}
D_B(x_i, x_j) &= \frac{|[x_i]_B^{\trianglerighteq c}\cup[x_j]_B^{\trianglerighteq}|}{|U|}\\
&= \frac{|(U-[x_i]_B^{\trianglerighteq})\cup[x_j]_B^{\trianglerighteq}|}{|U|}\\
&\geqslant \frac{|(U-[x_i]_B^{\trianglerighteq})\cup[x_i]_B^{\trianglerighteq}|}{|U|}\\
&= \frac{|U|}{|U|}=1
\end{aligned}$$

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