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# A quantitative approach to reasoning about incomplete knowledge☆

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ABSTRACT

In this paper, we aim to present a quantitative approach to reasoning about incomplete information. The study is conducted in MEL, a minimal epistemic logic relating modal languages to uncertainty theories. The proposed approach leads to two types of epistemic truth degrees of a proposition. Some related properties are derived. By means of a more general probability distribution on the set of epistemic states, two randomized versions of epistemic truth degrees are obtained. The connection between the notion of local probabilistic epistemic truth degree and belief function is also established. Based upon the fundamental notion of the global epistemic truth degree, the notion of epistemic similarity degree is also proposed and a kind of pseudo-metric used for approximate reasoning in MEL is thus derived. The obtained results provide a useful supplement to the existing study in the sense that it offers a quantitative approach instead of the qualitative manner in the literature.



## 1. Introduction

Various types of formal logical tools have been developed for the representation and reasoning of knowledge, including classical logics (propositional logic and predicate logic), nonclassical logics (multiple-valued logic, fuzzy logic, modal logic, description logic, etc.) [1,2,6,10–13]. They have been widely used as a formalism for knowledge representation in artificial intelligence and an analysis tool in computer science [1,11].

Classical propositional logic [12,19], among others, offers an approach to reasoning bout knowledge and beliefs based on complete information. It consists of syntactic and semantic parts. Semantically, a valuation, which assigns 1 or 0 to any propositional variable, represents an agent's complete information about the real world. At the syntactic level, it is possible to express that certain propositions are known or believed. However, it is usually not expressive enough to state that some propositions are acknowledged as being unknown to an agent. That is, reasoning about incomplete information requires a type of more expressive language. In [3,4], a minimal epistemic logic (MEL for short) that makes it possible to reason about partial information provided by a logically sophisticated agent was proposed. In MEL, atomic formulae express epistemic

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attitudes about formulae expressed in another propositional language. An agent can express both beliefs and explicitly ignored facts by using modal formulae of depth 1. Semantically, incomplete knowledge about the real world possessed by an agent is presented by a non-empty subset of interpretations, one and only one of which is, according to this agent's beliefs, the actual state of the world. This is what is usually called an epistemic state. Recently, theoretical studies on MEL have progressed rapidly. In [5], a system MEL$^+$ which is an extension of MEL having $S5$ syntax was proposed. The semantics is not based on Kripke models with equivalence relations, but on pairs made of an interpretation (representing the real state of facts) and a non-empty set of possible interpretations (representing an epistemic state). In [9], a generalized possibilistic logic extending both possibilistic logic and MEL was introduced. Then in [10], two types of approaches for reasoning about ignorance in GPL were proposed, based on the idea of minimal specificity and on the notion of guaranteed possibility, respectively. Moreover, in [7,8], the authors have proposed to translate three-valued logics of incomplete information into MEL, provided that the third truth-value refers to the idea of unknown Boolean truth-value.

Note that MEL just provides a qualitative approach to reasoning about incomplete information in the sense that an agent can express both beliefs and explicitly ignored facts. It lacks a quantitative characterization. Here we wish to mention Prof. Wang for his promising study of quantitative logic [19,20]. In his research work, all the basic logic notions were graded and three types of approximate reasoning patterns in the framework of propositional logic systems were proposed. As far as MEL is concerned, owing to its propositional feature, some basic ideas underlying quantitative logic can be naturally employed. More precisely, if $E \not\models \Box\alpha$ holds for an epistemic state $E$, i.e., $E \not\subseteq [\alpha]$, then the agent may not assert $\alpha$, such a point does not reflect the quantitative manner. It seems more reasonable to assume that if $E$ has a larger overlap with $[\alpha]$, the agent may assert $\alpha$ to a larger degree, moreover, how to combine the truth degrees of a proposition relative to different epistemic states to reflect the global epistemic truth degree requires further investigation. Such a quantitative description and the existing qualitative reasoning can greatly improve the expressive power of MEL.

Motivated by such a consideration, we seek to introduce the notion of epistemic truth degree of any formula in MEL, from two different types of viewpoints: local viewpoint and global viewpoint. In local viewpoint, we mainly restrict our attention to the set of epistemic states related to the propositional variables contained in each formula, in this sense, epistemic truth degrees of different formula cannot be compared directly. On the contrary, in global viewpoint, we have a fixed set of epistemic states for all formulae in MEL. From both viewpoints, by employing a more general probability distribution on the set of epistemic states, two randomized versions of epistemic truth degrees are also obtained. A comparative study with belief function, quantitative logic is also performed.

The rest of the paper proceeds as follows: In Section 2, we mainly review some basic notions in MEL. In Section 3, we introduce the notion of epistemic truth degree from the local viewpoint and global viewpoint, respectively, and a comparative study between the proposed notion and truth degree in quantitative logic is performed. In Section 4, by grading the notion of logical equivalence, we also introduce the notion of epistemic similarity degree for any two formulae in MEL. We complete this paper with some concluding remarks, as shown in Section 5.

## 2. Preliminaries

### *2.1. A simple logic for reasoning bout knowledge*

In what follows, we briefly recall the minimal epistemic logic [3,4] (MEL for short) used for reasoning about incomplete knowledge.

#### *2.1.1. Syntax*

In MEL, a finite set of propositional variables $PV = \{p_1, p_2, \cdots, p_k\}$ is used. By using Boolean connectives $\neg$ and $\wedge$, the set of Boolean propositional logic formulae (denoted by $\mathcal{L}$) can be built. The proposed syntax in MEL is obtained by encapsulating propositional language inside a language equipped with a modality denoted by $\Box$. The intended purpose is to completely separate propositions in $\mathcal{L}$ referring to the real world and propositions that refer to an agent's epistemic state, where the symbol $\Box$ appears.

The atoms of MEL are obtained by adding the unary connective $\Box$ in front of all formulae in $\mathcal{L}$, that is, atomic formulae of MEL are of the form $\Box\alpha, \alpha \in \mathcal{L}$. The intended meaning of $\Box\alpha$ is that an agent knows (or believes) proposition $\alpha$ is true, semantically speaking, $\alpha$ holds in each possible world compatible with the agent's epistemic state.

The set of MEL-formulae, denoted by $\phi, \psi, \cdots$ forms a set of modal language $\mathcal{L}_\Box$ generated recursively from the set $At$ of atomic formulae, with the help of the same Boolean connectives $\neg$, $\wedge$: that is,

$\Box\alpha \in \mathcal{L}_\Box$, whenever $\alpha \in \mathcal{L}$; $\neg\phi \in \mathcal{L}_\Box, \phi \wedge \psi \in \mathcal{L}_\Box$, whenever $\phi, \psi \in \mathcal{L}_\Box$.

From the primitive connectives $\neg$ and $\wedge$, one can define disjunction $\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$ and the modality $\diamond\alpha = \neg\Box\neg\alpha$, where $\alpha \in \mathcal{L}$, in the usual way. In MEL, modalities $\Box$ and $\diamond$ only apply to PL-formulae, contrary to usual modal logics. The intended meaning of $\diamond\alpha$ is that an agent either believes $\alpha$ is true, or ignores whether $\alpha$ is true or not.

**Definition 1.** [4] The set of axioms in MEL consists of the following formulae:

(PL)

(i) $\phi \to (\psi \to \phi)$,
(ii) $(\phi \to (\psi \to \mu)) \to ((\phi \to \psi) \to (\phi \to \mu))$,

(iii) $(\neg\phi\to\neg\psi)\to(\psi\to\phi)$,

(K) $\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta)$,

(N) $\Box\alpha$, whenever $\vdash_{PL}\alpha$,

(D) $\Box\alpha\to\diamond\alpha$.

Rule:

(MP) If $\phi$, $\phi\to\psi$, then $\psi$.

The following results shed more light on the connection between propositional logic and MEL. We use $\alpha$, $\beta$, $\cdots$ to denote formulae in proposition logic while $\phi$, $\psi$, $\cdots$ to denote formulae in MEL.

**Proposition 1.** *[4]*

(i) $\vdash_{PL}\alpha\to\beta$ *if and only if* $\vdash_{MEL}\Box\alpha\to\Box\beta$,

(ii) $\vdash_{PL}\alpha\to\beta$ *if and only if* $\vdash_{MEL}\diamond\alpha\to\diamond\beta$,

(iii) $\vdash_{PL}\alpha\to\beta$ *if and only if* $\vdash_{MEL}\Box\alpha\to\diamond\beta$.

*2.1.2. Semantics*

In the semantic part of MEL, an epistemic state $E$ is represented by a subset of mutually exclusive propositional valuations. Each valuation $v$ represents a 'possible world' compatible with the epistemic state of the agent, and has the form $v$: $PV\to\{0, 1\}$. Let $\mathcal{V}$ denote the set of valuations in propositional logic. Sometimes, a valuation in propositional logic can also be represented as a $k$-dimensional vector. For $\alpha\in\mathcal{L}$, $\overline{\alpha}$ is the truth function obtained by substituting each atom $p_i$ in $\alpha$ by $x_i$ and interpreting $\neg$ as the unary operation $\neg x=1-x$ on $\{0, 1\}$, and $\wedge$ as the corresponding minimum operation on $\{0, 1\}^2$. For any propositional valuation $v\in\mathcal{V}$ and $\alpha(p_1, \cdots, p_k)$, we have $v(\alpha)=\overline{\alpha}(v(p_1),\cdots,v(p_k))$. According to the definition of an epistemic state, we have $E\subseteq\mathcal{V}$. An epistemic state is further assumed to be non-empty (otherwise the agent is inconsistent).

The satisfaction of MEL-formulae is defined recursively, as follows. For any $\alpha\in\mathcal{L}, \phi, \psi\in\mathcal{L}_\Box, E(\neq\emptyset)\subseteq\mathcal{V}$:

- $E\models\Box\alpha$ if and only if $E\subseteq[\alpha]$,
- $E\models\neg\phi$ if and only if $E\not\models\phi$,
- $E\models\phi\wedge\psi$ if and only if $E\models\phi$ and $E\models\psi$, where $[\alpha]$ symbols the set of valuations which assign 1 to $\alpha$.

Considering that $\diamond\alpha=\neg\Box\neg\alpha$, we have

$E\models\diamond\alpha$ if and only if $E\cap[\alpha]\neq\emptyset$.

Some notions such as semantic equivalence, semantic consequence, etc., can be presented in a similar way as in propositional logic.

Since $\mathcal{L}_\Box$ is a propositional language, we can also define a standard propositional semantics. A propositional model of a MEL-formula is an interpretation $v$ of $\mathcal{L}_\Box$, that is, a mapping from $At$ to $\{0, 1\}$. Let $\mathcal{V}_\Box$ be the set of such propositional valuations. It is clear that the propositional logic thus defined from atoms in $At$, using language $\mathcal{L}_\Box$, axioms $PL$ and inference rule MP is sound and complete with respect to this propositional semantics. For the logic MEL, one must restrict to standard interpretations $v$ that satisfy axioms $K$, $D$ and $N$, forming a subset of $\mathcal{V}_\Box$ denoted by $\mathcal{V}_{MEL}$.

More precisely, valuations in $\mathcal{V}_{MEL}$ obey the following conditions:

$v(\phi)=1$ for any tautology $\phi\in\mathcal{L}_\Box$,

(K) $v(\Box(\alpha\to\beta))\le v(\Box\alpha\to\Box\beta)$,

(N) $v(\Box\alpha)=1$, whenever $\vdash_{PL}\alpha$,

(D) $v(\Box\alpha)\le v(\diamond\alpha)$.

*2.1.3. Soundness and completeness theorem*

It has been shown in [4] that $\mathcal{V}_{MEL}$ is in one-to-one correspondence with the set of epistemic states $\{E:\emptyset\neq E\subseteq\mathcal{V}\}$, and that the epistemic semantics is equivalent to the standard propositional semantics restricted to $\mathcal{V}_{MEL}$. The soundness and completeness of MEL follows directly from the one of propositional logic.

**Proposition 2.** *[4]*

(i) $\mathcal{V}_{MEL}$ *is in one-to-one correspondence with the set of epistemic states* $\{E:\emptyset\neq E\subseteq\mathcal{V}\}$.

(ii) *Given a valuation* $v$ *satisfying MEL axioms, there exists an associated epistemic state* $E_v=\{w\in\mathcal{V}|\ g_v(\mathcal{V}\backslash\{w\})=0\}$, *where* $g_v$ *is the necessity measure defined by* $g_v([\alpha])=v(\Box\alpha)$, *such that* $v(\phi)=1$ *(in PL semantics) if and only if* $E_v\models\phi$ *(in MEL semantics).*

(iii) *Given an epistemic state* $E$, *there exists a valuation* $v_E$ *such that* $v_E(\phi)=1$ *(in PL semantics) if and only if* $E\models\phi$ *(in MEL semantics), where* $v_E$ *is defined as follows:* $v_E(\Box\alpha)=1$ *if* $E\subseteq[\alpha]$, *and 0 otherwise.*

(iv) *For any valuation* $v$ *satisfying MEL axioms,* $v=v_{E_v}$; *For any epistemic state* $E$, $E=E_{v_E}$.

For any set $H$ of formulae in propositional logic, let $\Box H=\{\Box\alpha\mid\alpha\in H\}$.

**Theorem 1.** *[4]* $\Box H\vdash_{MEL}\Box\beta$ *if and only if* $H\vdash_{PL}\beta$.

**Theorem 2.** *[4] (Soundness and completeness) For any set* $\Gamma\cup\{\phi\}$ *of MEL-formulae,*

$\Gamma\vdash_{MEL}\phi$ if and only if $\Gamma\models_{MEL}\phi$.

## 3. Epistemic truth degree of propositions in MEL

In this section, we aim to present a graded version of satisfaction of propositions in MEL. We begin with the following analysis of the expressive power of MEL.

### 3.1. MEL provides a qualitative approach to reasoning about incomplete information

MEL emerges as an attempt to bridge possibility theory and modal logic, two commonly used knowledge representation frameworks. It offers a logical grounding to uncertainty theories of incomplete information. In MEL, an agent can express beliefs and ignored facts by modal formulae of depth 1, not objective formulae.

One remarkable feature of MEL is that it just provides a qualitative approach to reasoning about incomplete knowledge, which can be observed from both syntactic part and semantic part of MEL. In syntactics, by using formula of the form $\Box\alpha$, the agent can express that (s)he believes $\alpha$. Or in other words, the agent can assert beliefs in proposition $\alpha$. However, under incomplete information, an agent may not assert her/his beliefs in some propositions, that is, $\Box\alpha$ may not hold. Then in MEL, by using $\diamond\alpha$, the agent expresses that $\alpha$ is possibly true.

We then observe that in MEL, an agent can only express complete beliefs (including beliefs that $\alpha$ is true and $\alpha$ is false), partial beliefs of a formula in propositional logic. This is a kind of ternary judgement. However, such a language cannot express that the agent believes a proposition to an extent, when (s)he does not have enough information to believe $\alpha$. $\diamond\alpha$ just provides a qualitative belief, it cannot embody the idea of epistemic grade. A more intended case is that for two propositions $\alpha$ and $\beta$, an agent believes $\alpha$ to a larger degree than $\beta$, when s(he) has partial information about both $\alpha$ and $\beta$.

Such a point can also be observed from the viewpoint of semantics. Recall that $\Box\alpha$ holds if and only if in all possible worlds compatible with what the agent believes, it is the case that $\alpha$ holds. Mathematically, $E \models \Box\alpha$ if and only if $E\subseteq[\alpha]$. Similarly, $E\models\diamond\alpha$ if and only if $E\cap[\alpha]\neq\emptyset$. We observe that the truth of $\Box\alpha, \diamond\alpha$ is closely related to the set-theoretic relationship between $E$ and $[\alpha]$. However, the complete inclusion and non-empty intersection are just two qualitative descriptions of two crisp sets. It lacks some quantitative analysis. To state it more clearly, let us consider two formulae $\alpha$, $\beta$, in the same epistemic state $E$. If $[\alpha]$ is partially included in $E$ to a larger degree than $[\beta]$ (numerically, $\frac{|E\cap[\alpha]|}{|E|} \geq \frac{|E\cap[\beta]|}{|E|}$), then a more plausible conclusion should be that the agent has a larger degree of belief in $\alpha$ than $\beta$.

### 3.2. Epistemic truth degree of formulae in MEL from a local viewpoint

The analysis in the previous subsection motivates us to present a graded version of epistemic attitude towards formulae in MEL. As we will see below, the definition of epistemic truth degree depends on the number of propositional variables contained in the given formula. This is what the term "local viewpoint" explicitly refers to. Moreover, such an approach is mainly conducted at the semantical level. In view of the soundness and completeness theorem, the semantical gradedness also leads to the graded version of syntactic part.

Let $\phi \in \mathcal{L}_\Box$, we use $\mathcal{V}^{\phi}_{MEL}$ to denote the set of valuations satisfying MEL axioms when restricted to the set of propositional variables contained in $\phi$ and use $\mathcal{V}^{\phi}$ to denote the set of Boolean valuations on the set of propositional variables contained in $\phi$.

**Proposition 3.** *Let $\phi \in \mathcal{L}_\Box$ be a formula containing n propositional variables, then $|\mathcal{V}^{\phi}_{MEL}|= 2^{2^n} - 1$.*

**Proof.** Taking into account Proposition 2, we have $|\mathcal{V}^{\phi}_{MEL}| = |\{E : \emptyset \neq E \subseteq \{0,1\}^n\}|$. Then the desired result immediately follows from the fact that $|\{E : \emptyset \neq E \subseteq \{0,1\}^n\}| = 2^{2^n} - 1$. □

**Definition 2.** Let $\phi \in \mathcal{L}_\Box$, define

$$\tau^{L}_{MEL}(\phi) = \frac{|\{v \in \mathcal{V}^{\phi}_{MEL} : v(\phi) = 1\}|}{|\mathcal{V}^{\phi}_{MEL}|},$$

then we call $\tau^{L}_{MEL}(\phi)$ the local epistemic truth degree of $\phi$.

Observe from Definition 2 that $\tau^{L}_{MEL}(\phi)$ indeed measures the portion of valuations in $\mathcal{V}_{MEL}$ assigning 1 to $\phi$. This leads to the notion of epistemic truth degree.

In view of the fact that there exists a one-to-one correspondence between the set of epistemic states and that of valuations satisfying MEL axioms, Definition 2 can be equivalently stated as follows:

**Proposition 4.** *Let $\phi \in \mathcal{L}_\Box$ be a formula containing n propositional variables, and P be an even probability distribution on the set of epistemic states contained in $\{0, 1\}^n$, then*

$$\tau^{L}_{MEL}(\phi) = \sum\{P(E) \mid \pi(E,\phi) = 1, \emptyset \neq E \subseteq \{0,1\}^n\} = \sum\{P(E)\pi(E,\phi) \mid \emptyset \neq E \subseteq \{0,1\}^n\},$$

*where $\pi(E,\phi) = 1$ if and only if $E\models\phi$, and 0 otherwise.*

**Proof.** Trivially. □

**Example 1.** Let $\phi = \Box p_1, \varphi = \diamond p_1, \psi = \Box(p_1 \vee p_2)$, compute the local epistemic truth degree of $\phi$, $\varphi$ and $\psi$.

**Solution** For $\phi$, since $\phi$ contains only one propositional variable $p_1$, i.e., $n = 1$, then we have from Proposition 3 that the cardinality of $\mathcal{V}^{\phi}_{MEL}$ is equal to $2^{2^1} - 1 = 3$. Since $\mathcal{V}^{\phi} = \{0, 1\}$, we have $\{E : \emptyset \neq E \subseteq \mathcal{V}^{\phi}\} = \{E_1, E_2, E_3\}$ with $E_1 = \{1\}, E_2 = \{0\}, E_3 = \{1, 0\}$. According to Proposition 2, we have altogether 3 valuations satisfying MEL axioms, i.e., $\nu_{E_1}, \nu_{E_2}, \nu_{E_3}$. It can be checked easily that $\nu_{E_1}(\Box p_1) = 1$ while $\nu_{E_2}(\Box p_1) = \nu_{E_3}(\Box p_1) = 0$. Then we have from Definition 2 that $\tau^L_{MEL}(\phi) = \frac{1}{3}$.

For $\varphi$, it can be checked easily that $\nu_{E_1}(\diamond p_1) = \nu_{E_3}(\diamond p_1) = 1$ while $\nu_{E_2}(\diamond p_1) = 0$. Then we have from Definition 2 that $\tau^L_{MEL}(\varphi) = \frac{2}{3}$.

Similarly, for $\psi$, since $\psi$ contains two propositional variables $p_1, p_2$, i.e., $n = 2$, then we have from Proposition 3 that the cardinality of $\mathcal{V}^{\psi}_{MEL}$ is equal to $2^{2^2} - 1 = 15$. It can be checked that $\{E : \emptyset \neq E \subseteq \mathcal{V}^{\psi}\} = \{E_1, E_2, \cdots, E_{15}\}$, where

$E_1 = \{(1, 1)\}, E_2 = \{(1, 0)\}, E_3 = \{(0, 1)\}, E_4 = \{(0, 0)\}, E_5 = \{(1, 1), (1, 0)\}, E_6 = \{(1, 1), (0, 1)\}, E_7 = \{(1, 1), (0, 0)\}, E_8 = \{(1, 0), (0, 1)\}, E_9 = \{(1, 0), (0, 0)\}, E_{10} = \{(0, 1), (0, 0)\}, E_{11} = \{(1, 1), (1, 0), (0, 1)\}, E_{12} = \{(1, 1), (1, 0), (0, 0)\}, E_{13} = \{(1, 1), (0, 1), (0, 0)\}, E_{14} = \{(1, 0), (0, 1), (0, 0)\}, E_{15} = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$. By using Proposition 3, we can show that $\nu_{E_i}(\psi) = 1$ for $i = 1, 2, 3, 5, 6, 8, 11$. That is, we have altogether 7 valuations satisfying MEL axioms, then we have from Definition 2 that $\tau^L_{MEL}(\psi) = \frac{7}{15}$.

Observe from Definition 2 and Proposition 3 that the epistemic truth degree of a formula containing $n$ propositional variables must be a fraction with the denominator $2^{2^n} - 1$. Then one natural question arises: for any fraction of the form $\frac{t}{2^{2^n}-1} (0 \leq t \leq 2^{2^n} - 1, t \in \mathbf{Z})$, does there exist some formula $\phi$ such that $\tau^L_{MEL}(\phi) = \frac{t}{2^{2^n}-1}$? The following proposition gives a positive answer to this question.

**Proposition 5.** *For any proper fraction of the form* $\frac{t}{2^{2^n}-1} (0 \leq t \leq 2^{2^n} - 1, t \in \mathbf{Z})$, *there exists some formula (say as* $\phi$*) in* $\mathcal{L}_\Box$ *such that* $\tau^L_{MEL}(\phi) = \frac{t}{2^{2^n}-1}$.

**Proof.** For any epistemic state $E \subseteq \{0, 1\}^n$, define

$$\delta_E = \Box\alpha_E \wedge \bigwedge_{w \in E} \diamond\alpha_w,$$

where $\alpha_E$ is the formula satisfying $[\alpha_E] = E$ and $\alpha_w = \bigwedge_{w(p)=1} p \wedge \bigwedge_{w(p)=0} \neg p$ is a formula characterizing $w$. □

It has been shown in [4] that the set of epistemic models of $\delta_E$ is equal to the singleton $\{E\}$. For integer $t$ satisfying $0 \leq t \leq 2^{2^n} - 1$, we choose arbitrarily $t$ epistemic states contained in $\{0, 1\}^n$, say as $E_1, E_2, \cdots, E_t$, then we define $\phi = \delta_{E_1} \vee \delta_{E_2} \vee \cdots \vee \delta_{E_t}$. It can be checked that the set of epistemic models of $\phi$ is $\{E_1, E_2, \cdots, E_t\}$. Moreover, considering the fact that there exists a one-to-one correspondence between epistemic states and valuations satisfying MEL axioms, we can thus conclude that there exists $t$ valuations in $\mathcal{V}_{MEL}$ assigning 1 to $\phi$. Then we have from Definition 2 that $\tau^L_{MEL}(\phi) = \frac{t}{2^{2^n}-1}$.

The following proposition provides a necessary and sufficient condition for $\tau^L_{MEL}(\phi) = 1$.

**Proposition 6.** *Let* $\phi \in \mathcal{L}_\Box$, *then* $\tau^L_{MEL}(\phi) = 1$ *if and only if* $\phi$ *is a theorem in MEL.*

**Proof.** "⇒: " Assume, without any loss of generality, that $\phi$ contains $n$ propositional variables, then $\tau^L_{MEL}(\phi) = 1$ implies, according to Definition 2, that $\nu(\phi) = 1$ holds for each valuation $\nu \in \mathcal{V}^{\phi}_{MEL}$. Considering the one-to-one correspondence between the set of valuations satisfying MEL axioms and that of epistemic states, we then conclude from Proposition 2 that $E \models \phi$ holds for each epistemic state, that is, $\phi$ is a valid formula, then it follows from Theorem 2 that $\phi$ is a theorem in MEL. □

"⇐: " Conversely, if $\phi$ is a theorem in MEL, then according to Theorem 2, $E \models \phi$ holds for each epistemic state, considering the one-to-one correspondence between the set of valuations satisfying MEL axioms and that of epistemic states, we obtain that $\nu(\phi) = 1$ holds for each $\nu \in \mathcal{V}^{\phi}_{MEL}$. Then we have from Definition 2 that $\tau^L_{MEL}(\phi) = 1$.

**Proposition 7.** *Let* $\phi, \psi \in \mathcal{L}_\Box$ *be two formulae both containing n propositional variables, then* $\vdash_{MEL} \phi \to \psi$ *implies* $\tau^L_{MEL}(\phi) \leq \tau^L_{MEL}(\psi)$.

**Proof.** Since both $\phi$, $\psi$ contain $n$ propositional variables, then according to Definition 2, we have $\tau^L_{MEL}(\phi) = \frac{|\{\nu \in \mathcal{V}^{\phi}_{MEL} : \nu(\phi)=1\}|}{|\mathcal{V}^{\phi}_{MEL}|} = \frac{|\{\nu \in \mathcal{V}^{\phi}_{MEL} : \nu(\phi)=1\}|}{2^{2^n}-1}, \tau^L_{MEL}(\psi) = \frac{|\{\nu \in \mathcal{V}^{\psi}_{MEL} : \nu(\psi)=1\}|}{|\mathcal{V}^{\psi}_{MEL}|} = \frac{|\{\nu \in \mathcal{V}^{\psi}_{MEL} : \nu(\psi)=1\}|}{2^{2^n}-1}$ and $\mathcal{V}^{\phi}_{MEL} = \mathcal{V}^{\psi}_{MEL}$. Since $\phi \to \psi$ is a theorem in MEL, then we have from Theorem 2 that $\nu(\phi) \leq \nu(\psi)$ holds for each $\nu \in \mathcal{V}^{\phi}_{MEL}$, and therefore, $\tau^L_{MEL}(\phi) \leq \tau^L_{MEL}(\psi)$. □

Proposition 7 yields immediately the following corollary.

**Corollary 1.** *Let* $\phi, \psi \in \mathcal{L}_\Box$ *be two formulae both containing n propositional variables, if* $\phi$ *and* $\psi$ *are logically equivalent, then* $\tau^L_{MEL}(\phi) = \tau^L_{MEL}(\psi)$.

**Remark 1.** If the condition "both $\phi$ and $\psi$ contain the same propositional variables" is withdrawn, then the desired result may not hold. Please see the following example:

Let $\phi = \Box p_1$, $\psi = \Box p_1 \wedge \Box(p_2 \to p_2)$, clearly, $\phi$ and $\psi$ are logically equivalent. We have from Example 1 that $\tau^L_{MEL}(\phi) = \frac{1}{3}$. For $\psi$, since $\psi$ contains two propositional variables $p_1$, $p_2$, then we have from Proposition 3 that the cardinality of $\mathcal{V}^{\psi}_{MEL}$ is equal to $2^{2^2} - 1 = 15$. Since $\mathcal{V}^{\psi} = \{(1,1), (1,0), (0,1), (0,0)\}$, we obtain $\{E : \emptyset \neq E \subseteq \mathcal{V}^{\psi}\} = \{E_1, E_2, \cdots, E_{15}\}$, as shown in Example 1. We have from Proposition 2 that $v_{E_i}(\psi) = 1$ for $i = 1, 2, 5$. That is, we have altogether 3 valuations satisfying MEL axioms, then we have from Definition 2 that $\tau^L_{MEL}(\psi) = \frac{|\{v \in \mathcal{V}^{\psi}_{MEL} : v(\psi)=1\}|}{|\mathcal{V}^{\psi}_{MEL}|} = \frac{3}{15} = \frac{1}{5}$, which shows that $\tau^L_{MEL}(\phi) \not\equiv \tau^L_{MEL}(\psi)$.

**Proposition 8.** *Let $\phi \in \mathcal{L}_\Box$, then $\tau^L_{MEL}(\neg\phi) = 1 - \tau^L_{MEL}(\phi)$.*

**Proof.** Clearly, $\mathcal{V}^{\neg\phi}{}_{MEL} = \mathcal{V}^{\phi}_{MEL}$. Then by definition, $\tau^L_{MEL}(\neg\phi) = \frac{|\{v \in \mathcal{V}^{\neg\phi}_{MEL} : v(\neg\phi)=1\}|}{|\mathcal{V}^{\neg\phi}_{MEL}|} = \frac{|\{v \in \mathcal{V}^{\phi}_{MEL} : v(\phi)=0\}|}{|\mathcal{V}^{\phi}_{MEL}|} = 1 - \frac{|\{v \in \mathcal{V}^{\phi}_{MEL} : v(\phi)=1\}|}{|\mathcal{V}^{\phi}_{MEL}|} = 1 - \tau^L_{MEL}(\phi)$. □

**Proposition 9.** *Let $\phi, \psi \in \mathcal{L}_\Box$ be two formulae both containing n propositional variables, then $\tau^L_{MEL}(\phi \vee \psi) = \tau^L_{MEL}(\phi) + \tau^L_{MEL}(\psi) - \tau^L_{MEL}(\phi \wedge \psi)$.*

**Proof.** Since $\phi, \psi \in \mathcal{L}_\Box$ both contain $n$ propositional variables, then $\mathcal{V}^{\phi\vee\psi}_{MEL} = \mathcal{V}^{\phi}_{MEL} = \mathcal{V}^{\psi}_{MEL} = 2^{2^n} - 1$. For the sake of discussion, we denote them by $\mathcal{V}_{MEL}$. By definition, we have $\tau^L_{MEL}(\phi \vee \psi) = \frac{|\{v \in \mathcal{V}_{MEL} : v(\phi\vee\psi)=1\}|}{|\mathcal{V}_{MEL}|} = \frac{|\{v \in \mathcal{V}_{MEL} : v(\phi\vee\psi)=1\}|}{2^{2^n}-1} = \frac{|\{v \in \mathcal{V}_{MEL} : v(\phi)=1\}|}{2^{2^n}-1} + \frac{|\{v \in \mathcal{V}_{MEL} : v(\psi)=1\}|}{2^{2^n}-1} - \frac{|\{v \in \mathcal{V}_{MEL} : v(\phi\wedge\psi)=1\}|}{2^{2^n}-1} = \tau^L_{MEL}(\phi) + \tau^L_{MEL}(\psi) - \tau^L_{MEL}(\phi \wedge \psi)$. □

**Remark 2.** As in Proposition 7, the precondition in Proposition 9 cannot be withdrawn, otherwise, the desired result may not hold.

The notion of epistemic truth degree can be generalized by means of more general probability distributions on the set of epistemic states.

**Definition 3.** Let $\phi \in \mathcal{L}_\Box$ and $P$ be a probability distribution on the set of epistemic states. Define $\tau^{LP}_{MEL}(\phi) = \Sigma_{E \subseteq \mathcal{V}^{\phi}}\{P(E) \mid E \models \phi\}$, then we call $\tau^{LP}_{MEL}(\phi)$ the local probabilistic epistemic truth degree of $\phi$.

For MEL-formula of the form $\Box\alpha$ with $\alpha \in \mathcal{L}$, we can present a graded version of Definition 3 by further employing the inclusion degree relative to any epistemic state.

**Definition 4.** Let $\Box\alpha \in \mathcal{L}_\Box$ and $P$ be a probability distribution on the set of epistemic states. Define

$$\tau^{LPG}_{MEL}(\Box\alpha) = \Sigma_{E \subseteq \mathcal{V}^{\Box\alpha}}\{P(E)\frac{|E \cap [\alpha]|}{|E|}\},$$

where $[\alpha] = \{v \in \mathcal{V}^{\Box\alpha} \mid v(\alpha) = 1\}$.

We have the following proposition stating the relationship between $\tau^{LP}_{MEL}(\phi)$ and $\tau^{LPG}_{MEL}(\phi)$.

**Proposition 10.**

1. $\tau^{LP}_{MEL}(\Box\alpha) \leq \tau^{LPG}_{MEL}(\Box\alpha)$,
2. $\tau^{LP}_{MEL}(\Box\alpha) = \tau^{LPG}_{MEL}(\Box\alpha)$ *if and only if $[\alpha]$ is a singleton.*

**Example 2.** Let $\phi = \Box(p_1 \vee p_2)$ and $P$ be an even probability distribution on the set of epistemic states contained in $\{0, 1\}^2$. Compute the local probabilistic epistemic truth degree of $\phi$.

**Solution** Since $\phi$ contains two propositional variables $p_1$, $p_2$, i.e., $n = 2$, then we have from Proposition 2 that the cardinality of $\mathcal{V}^{\phi}_{MEL}$ is equal to $2^{2^2} - 1 = 15$. Since $\mathcal{V}^{\phi} = \{(1,1), (1,0), (0,1), (0,0)\}$, we have $\{E : \emptyset \neq E \subseteq \mathcal{V}^{\phi}\} = \{E_1, E_2, \cdots, E_{15}\}$, as shown in Example 1. Denote $p_1 \vee p_2$ by $\alpha$, an easy verification shows that $[\alpha] = \{(1,0), (0,1), (1,1)\}$ and $\frac{|E_i \cap [\alpha]|}{|E_i|} = 1$ for $i = 1, 2, 3, 5, 6, 8, 11$. For the other epistemic states, $\frac{|E_4 \cap [\alpha]|}{|E_4|} = 0$, $\frac{|E_7 \cap [\alpha]|}{|E_7|} = \frac{|E_9 \cap [\alpha]|}{|E_9|} = \frac{|E_{10} \cap [\alpha]|}{|E_{10}|} = \frac{1}{2}$, $\frac{|E_{12} \cap [\alpha]|}{|E_{12}|} = \frac{|E_{13} \cap [\alpha]|}{|E_{13}|} = \frac{|E_{14} \cap [\alpha]|}{|E_{14}|} = \frac{2}{3}$, $\frac{|E_{15} \cap [\alpha]|}{|E_{15}|} = \frac{3}{4}$, then we have from Definition 8 that $\tau^{LPG}_{MEL}(\phi) = \frac{7 + 3\times\frac{1}{2} + 3\times\frac{2}{3} + \frac{3}{4}}{2^{2^2}-1} = \frac{3}{4}$.

*3.3. Epistemic truth degree in MEL vs truth degree in QL*

Recall that in the quantitative logic (QL for short) originated in [20], Wang and Zhou introduced the concept of the degree of the truth in the framework of so called many-valued propositional logic systems with the intention of measuring to what extent a given formula is true. Some further studies along this research line can be found in [21,25]. Since MEL is also a propositional logic with the formulae of the form $\Box\alpha$ being its atoms, a close relationship between epistemic truth degree in MEL and that in QL is thus expected. In what follows, we will examine this issue in detail.

**Definition 5.** [20] Let $\alpha(p_1, \cdots, p_n)$ be a formula in classical propositional logic, define

$$\tau_2(\alpha) = \frac{|\overline{\alpha}^{-1}(1)|}{2^n},$$

then we call $\tau_2(\alpha)$ the truth degree of $\alpha$.

The following results concerning the notion of truth degree in QL are summarized in [20].

**Proposition 11.** *The set of truth degrees of formulae in classical propositional logic is equal to* $\{\frac{m}{2^n}, n = 1, 2, \cdots, m = 0, 1, \cdots, 2^n\}$.

**Proposition 12.** *Let* $\alpha$, $\beta$ *be two formulae in classical propositional logic, then*

1. $\tau_2(\alpha) = 1$ if and only if $\alpha$ is a theorem (or equivalently, a tautology) in classical propositional logic; $\tau_2(\alpha) = 0$ if and only if $\alpha$ is a refutable formula (or equivalently, a contradiction) in classical propositional logic,
2. $\tau_2(\neg\alpha) = 1 - \tau_2(\alpha)$,
3. $\tau_2(\alpha \vee \beta) = \tau_2(\alpha) + \tau_2(\beta) - \tau_2(\alpha \wedge \beta)$,
4. $\vdash \alpha \to \beta$ implies that $\tau_2(\alpha) \leq \tau_2(\beta)$.

There exist similarities as well as differences between the notion of truth degree in quantitative logic and that of epistemic truth degree in MEL.

**Similarities:** (i) Both are proposed with the aim of measuring the extent to which any formula is true.

(ii) They enjoy similar properties. For instance, the (epistemic) truth degree of a formula is equal to 1 if and only if it is a theorem in classical propositional logic (MEL); the (epistemic) truth degree of a formula is equal to 0 if and only if it is a refutable formula, et al.

**Differences:** (i) The epistemic truth degree is proposed in the context of incomplete information while the notion of truth degree was introduced in the context of complete information.

(ii) The epistemic truth degrees of formulae in MEL are of the form $\frac{m}{2^{2^n}-1}(0 \leq m \leq 2^{2^n} - 1, m \in \mathbf{Z})$ while the truth degrees of formulae in QL enjoy the form $\frac{m}{2^n}(0 \leq m \leq 2^n, m \in \mathbf{Z})$.

(iii) In QL, some properties, e.g., $\tau_2(\alpha \vee \beta) = \tau_2(\alpha) + \tau_2(\beta) - \tau_2(\alpha \wedge \beta)$, hold without any restriction on $\alpha$, $\beta$. On the contrary, in MEL, $\tau_2(\phi \vee \psi) = \tau_2(\phi) + \tau_2(\psi) - \tau_2(\phi \wedge \psi)$ does not hold generally. Proposition 9 tells that when both $\phi$ and $\psi$ have the same set of propositional variables, then the desired property holds.

The following proposition establishes the relationship between the notion of truth degree in QL and that of epistemic truth degree in MEL.

**Proposition 13.** *Let* $\alpha$ *be a formula in propositional logic containing n propositional variables* $p_1, \cdots, p_n$. *If* $\tau_2(\alpha) = \frac{m}{2^n}$, *then* $\tau^L_{MEL}(\Box\alpha) = \frac{2^m-1}{2^{2^n}-1}$.

**Proof.** According to Definition 5, $\tau_2(\alpha) = \frac{m}{2^n}$ means that $|\overline{\alpha}^{-1}(1)| = m$, that is, there are altogether $m$ valuations assigning truth value 1 to $\alpha$. For $\overline{\alpha}^{-1}(1)$, there are $2^m - 1$ non-empty subsets, say as $E_1, \cdots, E_{2^m-1}$. Since $E_i \subseteq [\alpha] = \overline{\alpha}^{-1}$, $v_{E_i}(\Box\alpha) = 1$ holds according to Proposition 2, $i = 1, \cdots, 2^m - 1$. For epistemic state $E'$ which are not subset of $E$, we have $v_{E'}(\Box\alpha) = 0$, that is, $E_1, \cdots, E_{2^m-1}$ are exactly those epistemic states at which $\Box\alpha$ is true. Considering the one-to-one correspondence relationship between epistemic states and valuations satisfying MEL axioms, we conclude that there are altogether $2^m - 1$ valuations satisfying axioms, then by Definition 2, $\tau^L_{MEL}(\Box\alpha) = \frac{2^m-1}{2^{2^n}-1}$. □

Proposition 13 yields the following corollary.

**Corollary 2.** *Let* $\alpha$ *be a refutable formula in propositional logic, then we have* $\tau^L_{MEL}(\Box\alpha) = 0$; *and vice versa.*

### *3.4. Connection with belief function theory*

In this subsection, it will be shown that the notion of epistemic truth degree can be interpreted from the viewpoint of belief function [16].

Recall that a belief function is a mapping *Bel* from $2^U$ to the unit interval [0,1] and satisfies the following axioms:

(F1) $Bel(\emptyset) = 0$,

(F2) $Bel(U) = 1$,

(F3) For every positive integer $n$ and every collection $A_1, \cdots, A_n \subseteq U$, $Bel(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \Sigma_i Bel(A_i) - \Sigma_{i<j} Bel(A_i \cap A_j) + \cdots + (-1)^{n+1} Bel(A_1 \cap \cdots \cap A_n)$.

Axioms (F1) and (F2) may be considered as normalization conditions. Axiom (F3) is a weaker version of the commonly known additivity axiom of probability functions. It is referred to as the axiom of superadditivity.

A belief function can be equivalently defined by another mapping, $m : 2^U \to [0, 1]$, which is called a basic probability assignment and satisfies two axioms:

(M1) $m(\emptyset) = 0$,

(M2) $\Sigma_{X \subseteq U} m(X) = 1$.

A subset $A \subseteq U$ with $m(A) > 0$ is called a focal element. Using the basic probability assignment, belief of $Y$ can be expressed as:

(M3) $Bel(Y) = \Sigma_{X \subseteq Y} m(X)$.

The following proposition shows that the notion of epistemic truth degree can be interpreted in terms of belief function.

**Proposition 14.** *Let $\alpha \in \mathcal{L}$ be a formula containing n propositional variables, P be a probability distribution on the set of epistemic states contained in $\{0, 1\}^n$ and $\tau_{MEL}^{LP}$ be the probabilistic epistemic truth degree mapping. Define a mapping $Bel : 2^{\{0,1\}^n} \to [0, 1]$ by*

$$Bel(X) = \tau_{MEL}^{LP}(\Box\alpha_X), X \subseteq \{0, 1\}^n, \qquad (1)$$

*where $\alpha_X$ is the formula satisfying $[\alpha_X] = X$, then Bel is a belief function on $2^{\{0,1\}^n}$.*

**Proof.** It follows from (1) and Definition 3 that $Bel(X) = \tau_{MEL}^{LP}(\Box\alpha_X) = \Sigma\{P(E) : E \models \Box\alpha_X\} = \Sigma\{P(E) : E \subseteq X\}$, then we have from (M3) that *Bel* is a belief function on $2^{\{0,1\}^n}$. □

**Proposition 15.** *Let Bel be a a belief function on $2^{\{0,1\}^n}$. Define a mapping $\tau : \mathcal{L}_\Box \to [0, 1]$ by*

$$\tau(\Box\alpha) = Bel(X_\alpha), \alpha \in \mathcal{L}, \qquad (2)$$

*where $X_\alpha = [\alpha]$, then there exists a probability distribution P on the set of epistemic states contained in $\{0, 1\}^n$ such that $\tau$ coincides with $\tau_{MEL}^{LP}$ induced by P.*

**Proof.** Since *Bel* is a belief function, there exists a probability distribution $P$ (satisfying $P(\emptyset) = 0$) on the set of epistemic states contained in $\{0, 1\}^n$ such that $Bel(Y) = \Sigma_{X\subseteq Y}P(X)$. Then we have from (2) and (M3) that $\tau(\Box\alpha) = Bel(X_\alpha) = \Sigma\{P(Y) \mid Y \subseteq X_\alpha\} = \Sigma\{P(Y) \mid Y \models \Box\alpha\}$, which shows that $\tau$ is equal to $\tau_{MEL}^{LP}$ with the restriction on the atomic formulae in $\mathcal{L}_\Box$. □

Indeed, Proposition 15 holds for each MEL-formula obtained by using only the connective $\wedge$, we omit the detailed proof here.

### *3.5. A new definition of epistemic truth degree from the global viewpoint*

In the definition of epistemic truth degree, for a formula containing $n$ propositional variables, a set of epistemic states of the form $\{E|\emptyset \neq E\subseteq\{0, 1\}^n\}$ is taken into consideration. That is, the definition of epistemic truth degree depends on the number of propositional variables contained in the concerned formula. In other words, such a definition is given from a local viewpoint. As a consequence, the epistemic truth degrees of different formulae cannot be compared directly owing to the fact that their epistemic settings are different.

The above-mentioned fact motivates us to present a new definition of epistemic truth degree from a global manner. In what follows, we always assume that MEL is built upon a fixed set of $n$ propositional variables and we use $\mathcal{V}_{MEL}$ to denote the set of valuations satisfying MEL axioms.

**Definition 6.** Let $\phi \in \mathcal{L}_\Box$, define

$$\tau_{MEL}^{G}(\phi) = \frac{|v \in \mathcal{V}_{MEL} \mid v(\phi) = 1|}{2^{2^n} - 1},$$

then we call $\tau_{MEL}^{G}(\phi)$ the global epistemic truth degree of $\phi$.

Observe from Definition 6 that for different formulae, the proposed new definition of global epistemic truth degree measures the portion of valuations assigning 1 to $A$ in the same set $\mathcal{V}_{MEL}$. This is what the term "global" refers to.

**Example 3.** Let $\phi = \Box p_1$, compute the global epistemic truth degree of $\phi$.

**Solution** For valuation $v \in \mathcal{V}_{MEL}$, we have from Proposition 2 that there exists an epistemic state $E\subseteq\{0, 1\}^n$ such that $v = v_E$. If $v(\phi) = 1$, i.e., $v_E(\Box p_1) = 1$, then we have $E\subseteq[p_1]$. Moreover, since $[p_1] = \{(1, x_2, \cdots, x_n)\}$ with $x_i \in \{0, 1\}$, it can be easily computed that $|[p_1]| = 2^{n-1}$, then the number of epistemic states satisfying $E\subseteq[p_1]$ is equal to $2^{2^{n-1}}$, and thus, by Definition 6, we have $\tau_{MEL}^{G}(\phi) = \frac{2^{2^{n-1}}}{2^{2^n}-1} = \frac{1}{2^{2^{n-1}}+1}$.

A similar proof shows that the previously proved propositions for $\tau_{MEL}^{L}$ also hold for $\tau_{MEL}^{G}$. Moreover, $\tau_{MEL}^{G}$ enjoys a more desired property like that in probability theory.

**Proposition 16.** *Let $\phi, \psi \in \mathcal{L}_\Box$, then $\tau_{MEL}^{G}(\phi \vee \psi) = \tau_{MEL}^{G}(\phi) + \tau_{MEL}^{G}(\psi) - \tau_{MEL}^{G}(\phi \wedge \psi)$.*

**Proof.** Since for any valuation $v \in \mathcal{V}_{MEL}$, $v(\phi \vee \psi) = v(\phi) + v(\psi) - v(\phi \wedge \psi)$, then by Definition 6, we have $\tau_{MEL}^{G}(\phi \vee \psi) = \tau_{MEL}^{G}(\phi) + \tau_{MEL}^{G}(\psi) - \tau_{MEL}^{G}(\phi \wedge \psi)$. □

The set of global epistemic truth degrees enjoys the following form.

**Proposition 17.** *Let H denote the set of global epistemic truth degrees of formulae in MEL, then $H = \{\frac{k}{2^{2^n}-1} \mid k = 0, 1, \cdots, 2^{2^n} - 1\}$.*

**Proposition 18.** *Let $\phi, \varphi, \psi \in \mathcal{L}_\Box$, then*

(i) If $\tau_{MEL}^{G}(\phi) \geq \alpha, \tau_{MEL}^{G}(\phi \to \varphi) \geq \beta$, then $\tau_{MEL}^{G}(\varphi) \geq \alpha + \beta - 1$,
(ii) If $\tau_{MEL}^{G}(\phi \to \varphi) \geq \alpha, \tau_{MEL}^{G}(\varphi \to \psi) \geq \beta$, then $\tau_{MEL}^{G}(\phi \to \psi) \geq \alpha + \beta - 1$.

**Proof.**

(i) $\tau^G_{MEL}(\phi \to \varphi) = \tau^G_{MEL}(\neg\phi \vee \varphi) = 1 - \tau^G_{MEL}(\phi) + \tau^G_{MEL}(\varphi) - \tau^G_{MEL}(\neg\phi \wedge \varphi) \geq \beta$, and so, $\tau^G_{MEL}(\varphi) \geq \beta + \tau^G_{MEL}(\phi) - 1 \geq \alpha + \beta - 1$.
(ii) It can be proved similarly.

□

The proposed notion of global epistemic truth degree can be further generalized by using a more general probability distribution on the set of epistemic states, as in Definition 8.

**Definition 7.** Let $P$ be a probability distribution on the set of epistemic states contained in $\{0, 1\}^n$. Define

$$\tau^{GP}_{MEL}(\phi) = \Sigma_{E\subseteq\{0,1\}^n}\{P(E) \mid E \models \phi\},$$

then we call $\tau^{GP}_{MEL}(\phi)$ probabilistic global epistemic truth degree of $\phi$.

**Proposition 19.** *There exists a one-to-one correspondence between the set of probabilistic global epistemic truth degree functions on $\mathcal{L}_\Box$ and that of belief functions on $2^{\{0,1\}^n}$.*

**Proof.** Given an epistemic truth degree function $\tau^{GP}_{MEL}$, then by Definition 7, $\tau^{GP}_{MEL}$ corresponds to a probability distribution on the set of epistemic states, which is indeed the basic probability assignment $m$ in theory of belief functions. Consequently, $m$ uniquely determines a belief function on $2^{\{0,1\}^n}$. This shows that an epistemic truth degree function uniquely determines a belief function.

Conversely, given a belief function *Bel* on the set $2^{\{0,1\}^n}$, then *Bel* uniquely determines a basic probability assignment function $m$, which is indeed a general probability distribution. By Definition 7, such a probability distribution can lead to a truth degree function $\tau^{GP}_{MEL}$. □

## 4. Epistemic similarity degree between formulae in MEL

In MEL, an important result states that $\Box\alpha$ and $\Box\beta$ are semantically equivalent if and only if $\alpha$, $\beta$ are logically equivalent in propositional logic. That is, in context of incomplete information, an agent can assert belief in $\alpha$ and $\beta$ simultaneously if and only if $\alpha$, $\beta$ are logically equivalent in propositional logic. For instance, $\Box p_1$ and $\Box(p_1 \wedge (p_2 \to p_2))$. In this section, by grading the notion of logical equivalence, we aim to present graded versions of similarities between any two formulae in MEL.

In what follows, we adopt the global view of epistemic truth degree to introduce the notion of epistemic similarity degree.

**Definition 8.** Let $\phi, \varphi \in \mathcal{L}_\Box$, define

$$\xi_{MEL}(\phi, \varphi) = \tau^G_{MEL}((\phi \to \varphi) \wedge (\varphi \to \phi)),$$

and we then call $\xi_{MEL}(\phi, \varphi)$ the epistemic similarity degree between $\phi$ and $\varphi$.

**Proposition 20.** *Let $\phi, \varphi, \psi \in \mathcal{L}_\Box$,*

(i) $\xi_{MEL}(\phi, \varphi) = 1$ *if and only if* $\phi \approx \varphi$,
(ii) $\xi_{MEL}(\phi, \varphi) = 0$ *if and only if* $\phi \approx \neg\varphi$,
(iii) $\xi_{MEL}(\phi, \varphi) + \xi_{MEL}(\phi, \neg\varphi) = 1$,
(iv) $\xi_{MEL}(\phi, \varphi) + \xi_{MEL}(\varphi, \psi) \leq 1 + \xi_{MEL}(\phi, \psi)$.

**Proof.** Both (i) and (ii) trivially hold. □

(iii) According to Definitions 6 and 8, it suffices to show that for each $v \in \mathcal{V}_{MEL}$, either $v((\phi \to \varphi) \wedge (\varphi \to \phi)) = 1$ or $v((\phi \to \neg\varphi) \wedge (\neg\varphi \to \phi)) = 1$ holds. Indeed, we have $v(\phi)$, $v(\varphi) \in \{0, 1\}$ that there are altogether four cases: (1) $(v(\phi), v(\varphi)) = (1, 1)$, (2) $(v(\phi), v(\varphi)) = (0, 0)$, (3) $(v(\phi), v(\varphi)) = (0, 1)$, (4) $(v(\phi), v(\varphi)) = (1, 0)$. Clearly, in cases (1) and (2), $v((\phi \to \varphi) \wedge (\varphi \to \phi)) = 1$ holds, and in case (3) and (4), $v((\phi \to \neg\varphi) \wedge (\neg\varphi \to \phi)) = 1$ holds, as desired.

(iv) Denote
$X = \{v \in \mathcal{V}_{MEL} \mid v((\phi \to \varphi) \wedge (\varphi \to \phi)) = 1\}$,
$Y = \{v \in \mathcal{V}_{MEL} \mid v((\varphi \to \psi) \wedge (\psi \to \varphi)) = 1\}$,
$Z = \{v \in \mathcal{V}_{MEL} \mid v((\phi \to \psi) \wedge (\psi \to \phi)) = 1\}$.
Clearly, $X \cap Y \subseteq Z$. According to Definitions 6 and 8, (iv) can be equivalently written as

$$\frac{|X|}{2^{2^n} - 1} + \frac{|Y|}{2^{2^n} - 1} \leq 1 + \frac{|Z|}{2^{2^n} - 1},$$

Since $\frac{|X|}{2^{2^n}-1} + \frac{|Y|}{2^{2^n}-1} = \frac{|X\cup Y|}{2^{2^n}-1} + \frac{|X\cap Y|}{2^{2^n}-1}$, moreover, $\frac{|X\cup Y|}{2^{2^n}-1} \leq 1$ and $\frac{|X\cap Y|}{2^{2^n}-1} \leq \frac{|Z|}{2^{2^n}-1}$, then the desired result holds immediately.

**Example 4.** Compute the similarity degree of $\Box p_1$ and $\Box p_2$.

**Solution** Let $\phi = ((\Box p_1 \to \Box p_2) \wedge (\Box p_2 \to \Box p_1))$ and $E \subseteq \{0, 1\}^n$, it can be checked easily that $E \models \phi$ if and only if either $E \models \Box(p_1 \wedge p_2)$ or $E \models \neg\Box p_1 \wedge \neg\Box p_2$ holds.

(i) Suppose that the former, i.e., $E \models \Box(p_1 \wedge p_2)$, holds, then $E \subseteq \{(1, 1, x_3, \cdots, x_n) | x_3, \cdots, x_n \in \{0, 1\}\}$, since $|\{(1, 1, x_3, \cdots, x_n) \mid x_3, \cdots, x_n \in \{0, 1\}\}| = 2^{n-2}$, there are altogether $2^{2^{n-2}} - 1$ types of such epistemic states.

(ii) Suppose that the latter, i.e., $E \models \neg\Box p_1 \wedge \neg\Box p_2$, holds, then $E \not\subseteq \{(1, x_2, x_3, \cdots, x_n) \mid x_2, \cdots, x_n \in \{0, 1\}\}$ and $E \not\subseteq \{(x_1, 1, x_3, \cdots, x_n) \mid x_1, x_3, \cdots, x_n \in \{0, 1\}\}$, a trivial computation shows that there are altogether $2^{2^{n-1}+1} - 1 - 2^{2^{n-2}}$ types of such epistemic states.

Then, according to Definition 8, we have $\xi_{MEL}(\Box p_1, \Box p_2) = \tau^G_{MEL}(\phi) = \frac{2^{2^{n-1}+1} - 1 - 2^{2^{n-2}} + 2^{2^{n-2}} - 1}{2^{2^n} - 1} = \frac{2^{2^{n-1}+1} - 2}{2^{2^n} - 1}$.

Considering the fact that there exists a one-to-one correspondence between the set of epistemic states and the set of valuations satisfying MEL axioms, the notion of epistemic similarity degree can be equivalently stated as follows.

**Proposition 21.** *Let P be an even probability distribution on the set of epistemic states, that is, for each* $\emptyset \neq E \subseteq \{0, 1\}^n$, $P(E) = \frac{1}{2^{2^n} - 1}$, *then for any* $\phi, \psi \in \mathcal{L}_\Box$, $\xi_{MEL}(\phi, \psi) = \Sigma\{P(E) \mid E \models \phi \wedge \psi \text{ or } E \models \neg\phi \wedge \neg\psi\}$.

### 4.1. Epistemic similarity degree in MEL vs similarity degree in QL

Let $\alpha, \beta \in \mathcal{L}$, recall first $\alpha$ and $\beta$ are said to be logically equivalent if $v(\alpha) = v(\beta)$ holds for each propositional valuation $v$. In quantitative logic, this notion has been graded to introduce the degree of the similarity between $\alpha$ and $\beta$.

**Definition 9.** [19] Let $\alpha, \beta \in \mathcal{L}$, define

$$\xi(\alpha, \beta) = \tau_2((\alpha \to \beta) \wedge (\beta \to \alpha)),$$

then we call $\xi(\alpha, \beta)$ the similarity degree between $\alpha$ and $\beta$.

Observe from Definition 9 that for $\alpha, \beta \in \mathcal{L}$ containing $n$ propositional variables, $\xi(\alpha, \beta)$ is a fraction of the form $\frac{m}{2^n}$.

The following proposition states the relationship between the notion of epistemic similarity degree in MEL and that of similarity degree in QL.

**Proposition 22.** *Let* $\alpha, \beta \in \mathcal{L}$, *if* $\xi(\alpha, \beta) = \frac{m}{2^n}$, *then* $\xi_{MEL}(\Box\alpha, \Box\beta) \geq \frac{2^m - 1}{2^{2^n} - 1}$.

**Proof.** According to Definitions 9 and 5, $\xi(\alpha, \beta) = \frac{m}{2^n}$ implies that $|[(\alpha \to \beta) \wedge (\beta \to \alpha)]| = m$. Denote $U = [(\alpha \to \beta) \wedge (\beta \to \alpha)]$, then an easy verification shows that $U = (\overline{\alpha}^{-1}(1) \cap \overline{\beta}^{-1}(1)) \cup (\overline{\alpha}^{-1}(0) \cap \overline{\beta}^{-1}(0))$. Moreover, for each epistemic state contained in $U$, it can be checked that $E \models (\Box\alpha \to \Box\beta) \wedge (\Box\beta \to \Box\alpha)$. We can prove it by considering the following two cases: □

**Case 1.** $E \subseteq (\overline{\alpha}^{-1}(1) \cap \overline{\beta}^{-1}(1))$, then we have $E \models \Box\alpha, E \models \Box\beta$, consequently, $E \models (\Box\alpha \to \Box\beta) \wedge (\Box\beta \to \Box\alpha)$.

**Case 2.** $E \cap (\overline{\alpha}^{-1}(0) \cap \overline{\beta}^{-1}(0)) \neq \emptyset$, then we have $E \not\models \Box\alpha, E \not\models \Box\beta$, consequently, $E \models (\Box\alpha \to \Box\beta) \wedge (\Box\beta \to \Box\alpha)$.

The above argument shows that there exist at least $2^m - 1$ epistemic states at which $(\Box\alpha \to \Box\beta) \wedge (\Box\beta \to \Box\alpha)$ is true. In view of the fact that there is a one-to-one correspondence between the set of epistemic states and that of valuations satisfying MEL axioms, we conclude that there exist at least $2^m - 1$ valuations in $\mathcal{V}_{MEL}$ which assign truth value 1 to $(\Box\alpha \to \Box\beta) \wedge (\Box\beta \to \Box\alpha)$. Then by Definition 8, we have that $\xi(\Box\alpha, \Box\beta) \geq \frac{2^m - 1}{2^{2^n} - 1}$.

Proposition 22 yields the following corollary.

**Corollary 3.** *Let* $\alpha, \beta \in \mathcal{L}$, *if* $\xi(\alpha, \beta) = 1$, *then* $\xi_{MEL}(\Box\alpha, \Box\beta) = 1$.

**Proof.** Let $\xi(\alpha, \beta) = \frac{m}{2^n}$, then we have from $\xi(\alpha, \beta) = 1$ that $m = 2^n$, which, together with Proposition 22, implies that $\xi_{MEL}(\Box\alpha, \Box\beta) \geq \frac{2^m - 1}{2^{2^n} - 1} = \frac{2^{2^n} - 1}{2^{2^n} - 1} = 1$. Therefore, $\xi_{MEL}(\Box\alpha, \Box\beta) = 1$. □

**Proposition 23.** *Let* $\alpha \in \mathcal{L}$, *if* $\tau(\alpha) = \frac{m}{2^n}$, *then* $\xi_{MEL}(\Box\alpha, \Diamond\alpha) = \frac{2^m + 2^{2^n - m} - 2}{2^{2^n} - 1}$.

**Proof.** It can be proved in a similar way as in Proposition 22. □

In quantitative logic, the notion of similarity degree can induce a kind of pseudo-metric in the following way.

**Definition 10.** Let $\alpha, \beta \in \mathcal{L}$, define

$$\rho(\alpha, \beta) = 1 - \xi(\alpha, \beta).$$

Consequently, a kind of logic metric space $(F(S), \rho)$ is thus obtained. By means of the pesudo-metric $\rho$, Wang and Zhou then introduced and investigated approximate reasoning [19] from various points of view.

**Definition 11.** Let $\Gamma$ be a theory, i.e., $\Gamma \subseteq F(S)$, and $\beta \in F(S)$, $\varepsilon > 0$, if

$$\rho(\beta, D(\Gamma)) = \inf\{\rho(\beta, \alpha) \mid \alpha \in D(\Gamma)\} < \varepsilon,$$

we then call $\beta$ is a conclusion of $\Gamma$ with error less than $\varepsilon$.

In MEL, we can similarly define a notion of epistemic pseudo-metric on the set of logical formulae in MEL.

**Definition 12.** Let $\phi, \psi \in \mathcal{L}_\square$, define

$$\rho_{MEL}(\phi, \psi) = 1 - \xi_{MEL}(\phi, \psi).$$

Owing to Proposition 20, it can be checked that $\rho_{MEL}$ is indeed a pseudo-metric on the set of logical formulae in MEL.

**Definition 13.** Let $\Gamma$ be a theory, i.e., $\Gamma \subseteq \mathcal{L}_\square$, and $\phi \in \mathcal{L}_\square, \varepsilon > 0$, if

$$\rho_{MEL}(\phi, D(\Gamma)) = \inf\{\rho_{MEL}(\phi, \psi) \mid \psi \in D(\Gamma)\} < \varepsilon,$$

we then call $\phi$ is a conclusion of $\Gamma$ with error less than $\varepsilon$.

In what follows, we explore the relationship between approximation reasoning pattern in quantitative logic and that in MEL.

**Proposition 24.** *Let* $\Gamma \subseteq \mathcal{L}, \alpha \in \mathcal{L}$, *if* $\rho(\alpha, D(\Gamma)) < \varepsilon$, *then* $\rho_{MEL}(\square\alpha, D(\square\Gamma)) < 1 - \frac{2^{2^n(1-\varepsilon)}-1}{2^{2^n}-1}$, *where* $\square\Gamma = \{\square\beta \mid \beta \in \Gamma\}$.

**Proof.** If $\rho(\alpha, D(\Gamma)) < \varepsilon$, then by Definition 13, there exists a formulae $\beta \in D(\Gamma)$ such that $\rho_{MEL}(\alpha, \beta) < \varepsilon$, which together with Definition 12 implies that $\xi_{MEL}(\alpha, \beta) > 1 - \varepsilon$. Combining with Proposition 22, we obtain $\xi_{MEL}(\square\alpha, \square\beta) > \frac{2^{2^n(1-\varepsilon)}-1}{2^{2^n}-1}$. Moreover, it follows from $\beta \in D(\Gamma)$ (i.e., $\Gamma \vdash \beta$) and Theorem 1 that $\square\Gamma \vdash \square\beta$. And therefore, $\rho_{MEL}(\square\alpha, D(\square\Gamma)) < 1 - \frac{2^{2^n(1-\varepsilon)}-1}{2^{2^n}-1}$. □

Proposition 24 yields the following corollary.

**Corollary 4.** *Let* $\Gamma \subseteq F(S)$, $\alpha \in F(S)$, *if* $\rho(\alpha, D(\Gamma)) = 0$, *then* $\rho_{MEL}(\square\alpha, D(\square\Gamma)) = 0$.

**Proof.** If $\rho(\alpha, D(\Gamma)) = 0$, then $\rho_{MEL}(\alpha, D(\Gamma)) < \varepsilon$ holds for arbitrarily chosen positive number $\varepsilon$. By Proposition 24, we have $\rho_{MEL}(\square\alpha, D(\square\Gamma)) < 1 - \frac{2^{2^n(1-\varepsilon)}-1}{2^{2^n}-1}$, which, together with $\lim_{\varepsilon \to 0} 1 - \frac{2^{2^n(1-\varepsilon)}-1}{2^{2^n}-1} = 0$, implies immediately that $\rho_{MEL}(\square\alpha, D(\square\Gamma)) = 0$. □

## 5. Concluding remarks

In this paper, we have made a modest attempt to present a quantitative analysis of a kind of epistemic logic MEL used for reasoning about incomplete information. More precisely, by grading some basic notions in MEL, we have introduced the notion of epistemic truth degree in context of incomplete information, then based upon such a fundamental notion, some derived notions such as epistemic similarity degree and epistemic pseudo-metric are also investigated.

The obtained results in this paper are the natural generalizations of those in [17,18] in the sense that rough set theory is one commonly used tool to deal with incomplete information. That is, the present study is conducted in a more general setting.

Some interesting issues along this research line are worthy of further study. For instance, the proposed notion of epistemic truth degree can be generalized by using the notion of inclusion degree in [14,22,24,26]. What is more, how to apply MEL to concept learning [15,23] from the viewpoint of cognitive computing needs further investigation.

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