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Granular Matrix: A New Approach for Granular Structure Reduction and Redundancy Evaluation

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***Abstract*—Granular structure is a mathematical expression of knowledge in granular computing and a direct determinant of the data processing efficiency. To improve the efficiency of data processing, many scholars have studied the reduction of granular structure. The attribute reduction and the granular reduction are two types of reduction on different layers of a granular structure, with the latter being both an essential step for granular structure reduction and the foundation of the attribute reduction. Yet compared with the attribute reduction, the granular reduction has received less attention from scholars. Therefore, a fuzzy granular reduction theory and a granular matrix based on the fuzzy β-coverings is proposed in this work. The insufficiency of the existing granular reduction theory for fuzzy β-coverings is pointed out, and proper sufficient and necessary conditions for two fuzzy β-coverings generating the same upper and lower approximations are also given in this work. In addition, to reduce and evaluate a fuzzy β-covering, a novel reduction algorithm based on granular matrix is proposed for the first time. Also, since fuzzy covering reduction is NP-hard, a heuristic greedy algorithm is designed to obtain a reduct. Numerical experiments have shown that the redundancy rates of neighborhood granule sets induced by some big scale data sets exceed 99%, which indicates that the existing neighborhood granulation methods need to be urgently improved. Based on this, concise granular structures and much more efficient feature selection algorithms can be proposed in the future.**

***Index Terms*—Artificial intelligence, Granular computing, Fuzzy sets, Rough sets, Granular reduction, Granular matrix.**

I. Introduction

WHen analyzing and solving problems, human brains can break down complex problems into multiple sub-problems and solve them one by one. Granular computing can simulate this function. By expressing the knowledge units as granules, granular computing establishes a multi-layered granule structure before selecting the appropriate granular layer according to the actual problem and performing calculation and reasoning on it, thus finally obtains the appropriate solution to the problem [1]. Granular computing, a granular reasoning-based intelligent computing approach, has been widely used in artificial intelligence, knowledge extraction and data mining (the granular computing-based intelligent computing and intelligence system are referred to as granular intelligence in this paper). The key of granular computing lies in the establishment and reduction of the granule structure. Fuzzy sets [2] and rough sets [3], [4], which provide different granulation and reasoning methods, are two main branches of granular computing. The establishment and reduction of granule structures based on fuzzy rough sets can provide a theoretical framework for feature selection, classification, clustering, decision making and knowledge extraction, based on which a variety of efficient algorithms can be designed.

Covering rough sets [5], [6] and binary relation rough sets [7]–[9] are two main generalized rough sets, but in fact, there is no essential difference between them. Covering rough sets mainly study the granule structure and granular reasoning from the perspective of geometry, while the binary relation rough sets focus more on the properties and the axiomatization of rough approximation operators from the perspective of algebra. The binary relations are generally transformed into granule sets through predecessor or successor neighborhoods, which can still be regarded as coverings on some universes. Therefore, the granule structure can be established on the basis of coverings (referred to as the covering granule structure), which can be used to solve related problems of different generalized rough sets.

Covering granular structure is a mathematical frame for intelligent computing, its simplification determines the effectiveness and efficiency of granular computing. Therefore, how to define and eliminate redundant granules remains a key problem. Or in other words, under what conditions do two coverings have the same lower and upper approximations? Many scholars have carried out studies on this issue [10], [11], among whom the first author of this paper proposed the covering approximation space theory based on the covering rough sets, which includes three covering approximation spaces: dual approximation space, $\mathcal{M}$-approximation space and $\mathcal{N}$-approximation space. The simplifications of a granule structure [11] based on different approximation spaces are studied, including granular reduction and attribute reduction. (1) In terms of the granular reduction, it is worth noting that the existing granular reduction [10] (i.e. union reduction) is based

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on $\mathcal{M}$-approximation space, while the granular reductions of dual approximation space and $\mathcal{N}$-approximation space are more complex. The existing literature [11] has offered the corresponding algorithm, that can effectively reduce the granule structure of a single attribute, thereby simplifying the subsequent knowledge process. (2) In terms of the attribute reduction, as the classical attribute reduction methods, namely the discernibility matrix and dependency degree, can only be used to solve the attribute reduction of $\mathcal{N}$-approximation space, the existing attribute reductions [12]–[25] are actually based on $\mathcal{N}$-approximation space. To solve the attribute reduction problem of dual approximation space and $\mathcal{M}$-approximation space, a related family method was proposed by the first author of this paper. The computational complexity of the dual approximation space and the $\mathcal{M}$-approximation space is lower than that of the $\mathcal{N}$-approximation space, which indicates a lower time and space complexity of the related family algorithm [26] based on the dual approximation space and the $\mathcal{M}$-approximation space than that of the discernibility matrix algorithm and dependency degree algorithm based on the $\mathcal{N}$-approximation space. Feature selection is an important data preprocessing method that can effectively compress the data dimension and provide higher generalization performance for subsequent machine learning models. In summary, the granular structure based on the approximation spaces represents the most important theoretical framework in granular computing. And on this basis, a systematic and in-depth study of different generalized rough sets can greatly improve the effectiveness and efficiency of knowledge extraction and machine learning.

To further improve the ability of knowledge granules to express information, scholars have studied various fuzzy granule structures and their reductions. (1) In the aspect of fuzzy granule reduction, Ma [27] generalized the fuzzy covering rough sets to the Fuzzy β-Covering rough sets (β-FC), and relaxed the condition of fuzzy covering. Yang et al. [28], [29] studied the fuzzy union reduction based on the β-FC. (2) Numerous attribute reductions based on fuzzy granules have been done by scholars [14], [20], [30]–[41], and a series of novel and effective algorithms have been proposed. These studies are very instructive, yet still exist some problems. The β-FC granule structure is more complicated than the covering granule structure, and since the granule structure of the β-FC has not been established yet, the fuzzy union reduction and attribute reduction based on the β-FC are easy to get bogged down into misunderstanding. For example, Yang et al. [28], [29] claimed that the fuzzy union reduction is the granular reduction of several β-FC models, yet we point out in this paper that this reduction is not sufficient. In addition, due to the complexity of the existing generation process of fuzzy granules, a variety of fuzzy attribute reduction algorithms are inefficient, and the introduction of fuzzy β-approximation space can greatly improve this situation. In short, the root of these problems is the lack of basic theory of fuzzy granule space. Therefore, defining an approximation space based on the β-FC and establishing the granule reduction of β-FC can help to improve the effectiveness and efficiency of knowledge extraction and machine learning, and avoid misunderstanding or repeated research.

In view of the high granule redundancy rate pointed in this paper, a new granulation method is proposed to generate concise β-FCs. It is worth noting that there are only 50-100 fuzzy granules in a covering, while the existing granulation methods [12]–[20], [23], [26], [32]–[34], [36], [37], [40], [41] generate n granules (n represents the number of sample). For big data sets, there may be over a million granules in a covering, which results in a huge difference in computational efficiency. In conclusion, the ability of fuzzy granule to express information is very powerful, which allows us to use very few granules to express a big scale of data sets. Based on this advantage, feature selection and other machine learning processes can be greatly accelerated.

The rest of this paper is organized as follows. Some related notions and definitions are reviewed in Section II. Fuzzy neighborhood reduction theory based on fuzzy β-coverings is studied in Section III. The concept of granular matrix is proposed and studied in Section IV. In Section V, a heuristic algorithm to obtain a fuzzy $\mathcal{N}$-reduct is constructed. Experiments are conducted in Section VI. Section VII concludes the whole paper.

II. BACKGROUND

Firstly, basic notions related to fuzzy β-covering rough sets are introduced as below.

Definition 1. *[27], [29](Fuzzy β-covering) Given a set of objects U, $\mathcal{F}(U)$ is the collection of all fuzzy sets defined on U. For each $\beta \in (0,1]$, $\widehat{\mathscr{C}} = \{\widehat{C}_1, \widehat{C}_2, \ldots, \widehat{C}_m\}$, where $\widehat{C}_i \in \mathcal{F}(U)(i = 1, 2, \ldots, m)$, is defined as a Fuzzy β-Covering (β-FC) of U if $(\bigcup_{i=1}^{m} \widehat{C}_i)(x) \geq \beta$ for each $x \in U$. $(U, \widehat{\mathscr{C}})$ is referred to as a Fuzzy β-Covering approximation Space (β-FCSpace). For each $x \in U$, the fuzzy β-neighborhood $(\widetilde{N}_x^{\beta})_{\widehat{\mathscr{C}}}$, fuzzy complementary β-neighborhood $(\widetilde{M}_x^{\beta})_{\widehat{\mathscr{C}}}$ and fuzzy β-minimal description $(\widetilde{Md}_x^{\beta})_{\widehat{\mathscr{C}}}$ of x are defined as:*

$$(\widetilde{N}_x^{\beta})_{\widehat{\mathscr{C}}} = \bigcap\{\widehat{C}_i \in \widehat{\mathscr{C}} : \widehat{C}_i(x) \geq \beta\}.$$

$$(\widetilde{M}_x^{\beta})_{\widehat{\mathscr{C}}}(y) = (\widetilde{N}_y^{\beta})_{\widehat{\mathscr{C}}}(x) \text{ for all } y \in U.$$

$$(\widetilde{Md}_x^{\beta})_{\widehat{\mathscr{C}}} = \{\widehat{C} \in \widehat{\mathscr{C}} : \widehat{C}(x) \geq \beta, \text{ for any } \widehat{D} \in \widehat{\mathscr{C}}, \text{ if } \widehat{D}(x) \geq \beta \text{ and } \widehat{D} \subseteq \widehat{C}, \text{ then } \widehat{C} = \widehat{D}\}.$$

We call $\Theta^{\beta}(\widehat{\mathscr{C}}) = \{\widetilde{N}_x^{\beta} : x \in U\}$ a fuzzy β-neighborhood family, $\overline{\Theta}^{\beta}(\widehat{\mathscr{C}}) = \{\widetilde{M}_x^{\beta} : x \in U\}$ a fuzzy complementary β-neighborhood family and $\Delta^{\beta}(\widehat{\mathscr{C}}) = \{\widetilde{Md}_x^{\beta} : x \in U\}$ a fuzzy β-minimal description family generated by $\widehat{\mathscr{C}}$ (where $\widehat{\mathscr{C}}$ can be ignored as long as there is no confusion).

When describing an object, generally not all characteristics are of equal importance. To eliminate redundant information and retain its valuable characteristics, notions of fuzzy β-neighborhood, fuzzy complementary β-neighborhood and fuzzy β-minimal description are proposed. The remaining essential characteristics constitute the new β-FCs of U, which are called fuzzy β-neighborhood family, fuzzy complementary β-neighborhood family and fuzzy β-minimal description family.

Based on the notions of fuzzy β-neighborhoods and fuzzy complementary β-neighborhoods, scholars have proposed four kinds of β-FC models [27], [29], which are listed as below.

Definition 2. *[27], [29] Given a β-FCSpace $(U, \widehat{\mathscr{C}})$, the lower approximations $\widetilde{P^-}(X)$, $\widetilde{FL}(X)$, $\widetilde{SL}(X)$, $\widetilde{TL}(X)$ and upper approximations $\widetilde{P^+}(X)$, $\widetilde{FH}(X)$, $\widetilde{SH}(X)$, $\widetilde{TH}(X)$ of $X \in \mathcal{F}(U)$ are defined as (for any $x \in U$):*

$\widetilde{P^-}(X)(x) = \wedge_{y\in U}[(1 - \widetilde{N}_x^\beta(y)) \vee X(y)]$;
$\widetilde{P^+}(X)(x) = \vee_{y\in U}[\widetilde{N}_x^\beta(y) \wedge X(y)]$.
$\widetilde{FL}(X)(x) = \wedge_{y\in U}[(1 - \widetilde{M}_x^\beta(y)) \vee X(y)]$;
$\widetilde{FH}(X)(x) = \vee_{y\in U}[\widetilde{M}_x^\beta(y) \wedge X(y)]$.
$\widetilde{SL}(X)(x) = \wedge_{y\in U}[(1 - \widetilde{N}_x^\beta(y)) \vee (1 - \widetilde{M}_x^\beta(y)) \vee X(y)]$;
$\widetilde{SH}(X)(x) = \vee_{y\in U}[\widetilde{N}_x^\beta(y) \wedge \widetilde{M}_x^\beta(y) \wedge X(y)]$.
$\widetilde{TL}(X)(x) = \wedge_{y\in U}[((1 - \widetilde{N}_x^\beta(y)) \wedge (1 - \widetilde{M}_x^\beta(y))) \vee X(y)]$;
$\widetilde{TH}(X)(x) = \vee_{y\in U}[(\widetilde{N}_x^\beta(y) \vee \widetilde{M}_x^\beta(y)) \wedge X(y)]$.

These four β-FC models are called the primal ($\widetilde{P^-}$ and $\widetilde{P^+}$), the first ($\widetilde{FL}$ and $\widetilde{FH}$), the second ($\widetilde{SL}$ and $\widetilde{SH}$) and the third ($\widetilde{TL}$ and $\widetilde{TH}$) β-FC model in this paper, respectively. Although their approximation elements and approximation ways are different, the propositions in following section indicate that their reduction methods are the same.

Definition 3. *[29](Fuzzy union reduct) For a given β-FC $\widehat{\mathscr{C}}$ of U and $\widehat{C} \in \widehat{\mathscr{C}}$, $\widehat{C}$ is called a fuzzy union reducible element of $\widehat{\mathscr{C}}$ if $\widehat{C}$ is the union of some fuzzy sets in $\widehat{\mathscr{C}} - \{\widehat{C}\}$, or else $\widehat{C}$ is a fuzzy union irreducible element of $\widehat{\mathscr{C}}$. The collection of all fuzzy union reducible elements and the collection of all fuzzy union irreducible elements of $\widehat{\mathscr{C}}$ are denoted by $FUR(\widehat{\mathscr{C}})$ and $FUI(\widehat{\mathscr{C}})$ respectively. For any $\widehat{\mathscr{D}} \subseteq \widehat{\mathscr{C}}$, supposing $FUR(\widehat{\mathscr{C}}) = \widehat{\mathscr{C}} - \widehat{\mathscr{D}}$, then $\widehat{\mathscr{D}}$ is called a fuzzy union reduct of $\widehat{\mathscr{C}}$. We denote it as $\Gamma(\widehat{\mathscr{C}})$.*

A reducible element is defined as fuzzy union reducible in [29]. However, even if we delete all fuzzy union reducible elements from a β-FC based on the β-FC models in [29], there may still be some superfluous elements, as shown in the example below.

Example 1. *Considering the set of objects $U = \{x_1, x_2, x_3, x_4\}$, $\widehat{\mathscr{C}} = \{\widehat{C}_1, \widehat{C}_2, \widehat{C}_3, \widehat{C}_4, \widehat{C}_5, \widehat{C}_6, \widehat{C}_7\}$ is a β-FC for $\beta = 0.5$. The fuzzy blocks are listed below.*

$\widehat{C}_1 = \frac{0.7}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4}$,
$\widehat{C}_2 = \frac{0.2}{x_1} + \frac{0.7}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4}$,
$\widehat{C}_3 = \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.3}{x_4}$,
$\widehat{C}_4 = \frac{0.2}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4}$,
$\widehat{C}_5 = \frac{0.4}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.3}{x_4}$,
$\widehat{C}_6 = \frac{0.7}{x_1} + \frac{0.9}{x_2} + \frac{0.8}{x_3} + \frac{0.9}{x_4}$,
$\widehat{C}_7 = \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4}$.

Based on Definition 1, the fuzzy β-neighborhood family of $\widehat{\mathscr{C}}$ is calculated as follows:

$(\widetilde{N}_{x_1}^{0.5})_{\widehat{\mathscr{C}}} = \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4}$,
$(\widetilde{N}_{x_2}^{0.5})_{\widehat{\mathscr{C}}} = \frac{0.2}{x_1} + \frac{0.7}{x_2} + \frac{0.3}{x_3} + \frac{0.3}{x_4}$,
$(\widetilde{N}_{x_3}^{0.5})_{\widehat{\mathscr{C}}} = \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.3}{x_4}$,
$(\widetilde{N}_{x_4}^{0.5})_{\widehat{\mathscr{C}}} = \frac{0.2}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4}$.

The fuzzy complementary β-neighborhood family of $\widehat{\mathscr{C}}$ is calculated as follows:

$(\widetilde{M}_{x_1}^{0.5})_{\widehat{\mathscr{C}}} = \frac{0.6}{x_1} + \frac{0.2}{x_2} + \frac{0.6}{x_3} + \frac{0.2}{x_4}$,
$(\widetilde{M}_{x_2}^{0.5})_{\widehat{\mathscr{C}}} = \frac{0.1}{x_1} + \frac{0.7}{x_2} + \frac{0.1}{x_3} + \frac{0.8}{x_4}$,
$(\widetilde{M}_{x_3}^{0.5})_{\widehat{\mathscr{C}}} = \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.3}{x_4}$,
$(\widetilde{M}_{x_4}^{0.5})_{\widehat{\mathscr{C}}} = \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.7}{x_4}$.

We can get $\Theta^{0.5}(\widehat{\mathscr{C}}) = \Theta^{0.5}(\widehat{\mathscr{C}} - \{C_1\}) = \Theta^{0.5}(\widehat{\mathscr{C}} - \{C_5\}) = \Theta^{0.5}(\widehat{\mathscr{C}} - \{C_6\}) = \Theta^{0.5}(\widehat{\mathscr{C}} - \{C_7\})$, while $\Theta^{0.5}(\widehat{\mathscr{C}}) \neq \Theta^{0.5}(\widehat{\mathscr{C}} - \{C_2\})$, $\Theta^{0.5}(\widehat{\mathscr{C}}) \neq \Theta^{0.5}(\widehat{\mathscr{C}} - \{C_3\})$ and $\Theta^{0.5}(\widehat{\mathscr{C}}) \neq \Theta^{0.5}(\widehat{\mathscr{C}} - \{C_4\})$. Since any fuzzy complementary β-neighborhood is determined by the corresponding fuzzy β-neighborhood, it is apparently that: $\overline{\Theta}^{0.5}(\widehat{\mathscr{C}}) = \overline{\Theta}^{0.5}(\widehat{\mathscr{C}} - \{C_1\}) = \overline{\Theta}^{0.5}(\widehat{\mathscr{C}} - \{C_5\}) = \overline{\Theta}^{0.5}(\widehat{\mathscr{C}} - \{C_6\}) = \overline{\Theta}^{0.5}(\widehat{\mathscr{C}} - \{C_7\})$, while $\overline{\Theta}^{0.5}(\widehat{\mathscr{C}}) \neq \overline{\Theta}^{0.5}(\widehat{\mathscr{C}} - \{C_2\})$, $\overline{\Theta}^{0.5}(\widehat{\mathscr{C}}) \neq \overline{\Theta}^{0.5}(\widehat{\mathscr{C}} - \{C_3\})$ and $\overline{\Theta}^{0.5}(\widehat{\mathscr{C}}) \neq \overline{\Theta}^{0.5}(\widehat{\mathscr{C}} - \{C_4\})$.

Considering that for the primal, the first, the second, the third β-FC rough set models, if the $\Theta^{0.5}(\widehat{\mathscr{C}})$ and $\overline{\Theta}^{0.5}(\widehat{\mathscr{C}})$ are invariant, then the corresponding upper and lower approximations will not change. As a result, the set of all irreducible elements is $\{\widehat{C}_2, \widehat{C}_3, \widehat{C}_4\}$ and the set of all reducible elements is $\{\widehat{C}_1, \widehat{C}_5, \widehat{C}_6, \widehat{C}_7\}$. However, none of the reducible elements is fuzzy union reducible. Thus, only reducing fuzzy union reducible elements from a β-FC is far from reduced, we need to find other types of reducible elements.

III. Fuzzy Neighborhood Reduction Theory

Under what condition do two β-FCs have the same β-FC lower and upper approximations of an arbitrary fuzzy set? This is an important question for rough set theory and data mining. Yang et al. [29] claimed that they had found the necessary and sufficient conditions for this question, that is, the β-FC lower and upper approximations of two β-FCs are the same if and only if their fuzzy union reducts are the same. As shown in Example 1, the fuzzy union reducts of $\widehat{\mathscr{C}}$ and $\widehat{\mathscr{C}} - \{C_1\}$ are different, but they have the same lower and upper approximations of an arbitrary fuzzy set. Obviously, this condition is not necessary and needs to be revised as below.

Proposition 1. *Given two different β-FCs $\widehat{\mathscr{C}_1}$ and $\widehat{\mathscr{C}_2}$ of U, then $\widehat{\mathscr{C}_1}$,$\widehat{\mathscr{C}_2}$ generate the same primal, first, second and third type of β-FC upper and lower approximations of $X \in \mathcal{F}(U)$ if $\Gamma(\widehat{\mathscr{C}_1}) = \Gamma(\widehat{\mathscr{C}_2})$, but not vice versa.*

This proposition suggests that there must be other kinds of reducible elements besides fuzzy union reducible elements. Thus, an appropriate method should be constructed to reduce β-FCs. It is worth noting that the fuzzy β-neighborhood family and fuzzy complementary β-neighborhood family are the collection of all approximation elements applied to the primal, the first, the second and the third β-FC models, which means that the lower and upper approximations of an arbitrary fuzzy set will not change as long as the fuzzy β-neighborhood family and fuzzy complementary β-neighborhood family of a β-FC remain the same. Evidently, the fuzzy β-neighborhood family and fuzzy complementary β-neighborhood family are the key notions for the β-FC reduction. Thus, the granular reduction based on the primal, the first, the second and the third β-FC models is refered to as fuzzy neighborhood reduction theory (or fuzzy $\mathcal{N}$-reduction theory for short).

To reveal the relation between a fuzzy β-neighborhood family and the related fuzzy complementary β-neighborhood family, their matrix representations are defined.

Definition 4. *For a given β-FC $\widehat{\mathscr{C}}$ of U, $\Theta^{\beta}(\widehat{\mathscr{C}}) = \{\widetilde{N}^{\beta}_{x_i} : x_i \in U\}$ and $\overline{\Theta}^{\beta}(\widehat{\mathscr{C}}) = \{\widetilde{M}^{\beta}_{x_i} : x_i \in U\}$ are the fuzzy β-neighborhood family and the fuzzy complementary β-neighborhood family, respectively. $M(\Theta^{\beta}(\widehat{\mathscr{C}})) = (a_{ij})_{n\times n}$, where $a_{ij} = \widetilde{N}^{\beta}_{x_i}(x_j)$ is a matrix representation of $\Theta^{\beta}(\widehat{\mathscr{C}})$. In the same way, $M(\overline{\Theta}^{\beta}(\widehat{\mathscr{C}})) = (b_{ij})_{n\times n}$, where $b_{ij} = \widetilde{M}^{\beta}_{x_i}(x_j)$ is a matrix representation of $\overline{\Theta}^{\beta}(\widehat{\mathscr{C}})$.*

Proposition 2. *$(M(\Theta^{\beta}(\widehat{\mathscr{C}})))^T = M(\overline{\Theta}^{\beta}(\widehat{\mathscr{C}}))$, where $(M(\Theta^{\beta}(\widehat{\mathscr{C}})))^T$ denotes the transpose of $M(\Theta^{\beta}(\widehat{\mathscr{C}}))$.*

The following proposition reveals the relation between fuzzy β-neighborhood family and fuzzy complementary β-neighborhood family based on Definition 1 and Proposition 2.

Proposition 3. *Given two different β-FCs $\widehat{\mathscr{C}_1}$ and $\widehat{\mathscr{C}_2}$ of U, then $\Theta^{\beta}(\widehat{\mathscr{C}_1}) = \Theta^{\beta}(\widehat{\mathscr{C}_2})$ if and only if $\overline{\Theta}^{\beta}(\widehat{\mathscr{C}_1}) = \overline{\Theta}^{\beta}(\widehat{\mathscr{C}_2})$.*

Considering that the lower and upper approximations of β-FCs are determined by the fuzzy β-neighborhood family and the fuzzy complementary β-neighborhood family, while the fuzzy complementary β-neighborhood family is determined by the fuzzy β-neighborhood family, then the lower and upper approximations of β-FCs can only be determined by the fuzzy β-neighborhood family.

For fuzzy granular reduction, although there is only one fuzzy union reduct for each β-FC, there may be more than one fuzzy $\mathcal{N}$-reduct of a β-FC, indicating that the granular reduction about fuzzy β-neighborhood family is much more complex than the fuzzy union reduction. In this section, definitions are first given to a fuzzy $\mathcal{N}$-reducible element, a fuzzy $\mathcal{N}$-irreducible β-FC and a fuzzy $\mathcal{N}$-reduct based on fuzzy β-neighborhood family. Meanwhile, the characteristics of fuzzy $\mathcal{N}$-reduction are studied and the relation between the fuzzy union reduction and the fuzzy $\mathcal{N}$-reduction is examined. On this basis, a new granular reduction algorithm based on granular matrix is initially proposed to obtain all fuzzy $\mathcal{N}$-reducts of a β-FC.

Definition 5. *(Fuzzy $\mathcal{N}$-reducible element) For a given β-FC $\widehat{\mathscr{C}}$ of U and $\widehat{C} \in \widehat{\mathscr{C}}$, $\Theta^{\beta}(\widehat{\mathscr{C}})$ is the fuzzy β-neighborhood family. $\widehat{C}$ is called a fuzzy $\mathcal{N}$-reducible element of $\widehat{\mathscr{C}}$ if $\Theta^{\beta}(\widehat{\mathscr{C}}) = \Theta^{\beta}(\widehat{\mathscr{C}} - \{\widehat{C}\})$, or else $\widehat{C}$ is called a fuzzy $\mathcal{N}$-irreducible element of $\widehat{\mathscr{C}}$. The collection of all fuzzy $\mathcal{N}$-reducible elements and the collection of all fuzzy $\mathcal{N}$-irreducible elements of $\widehat{\mathscr{C}}$ are denoted by $FNR(\widehat{\mathscr{C}})$ and $FNI(\widehat{\mathscr{C}})$ respectively.*

Since the fuzzy β-neighborhood family is the collection of all approximation elements in the primal, first, second and third type of β-FC models, it is defined as the fuzzy $\mathcal{N}$-approximation space in this paper. Hence, a reduct of a β-FC is the minimal subsets that keeps the fuzzy β-neighborhood family invariant.

The following definition describes a special block of a β-FC that has no contribution to the β-FC.

Definition 6. *(Null element) For a given β-FC $\widehat{\mathscr{C}}$ of U and $\widehat{C} \in \widehat{\mathscr{C}}$, $\widehat{C}$ is called a null element of $\widehat{\mathscr{C}}$ if $\widehat{C}(x) < \beta$ for all $x \in U$.*

It is obvious that a null element is a fuzzy $\mathcal{N}$-reducible element for any β-FC.

Proposition 4. *For a given β-FC $\widehat{\mathscr{C}}$ of U and $\widehat{C} \in \widehat{\mathscr{C}}$, suppose $\widehat{C}$ is a null element of $\widehat{\mathscr{C}}$, then $\widehat{C}$ is a fuzzy $\mathcal{N}$-reducible element.*

Proof. It is evident from the definitions of null element and fuzzy β-neighborhood description. □

Definition 7. *(Fuzzy $\mathcal{N}$-irreducible β-FC) For a given β-FC $\widehat{\mathscr{C}}$ of U, $\Theta^{\beta}(\widehat{\mathscr{C}})$ is the fuzzy β-neighborhood family, $\widehat{\mathscr{C}}$ is fuzzy $\mathcal{N}$-irreducible if each $\widehat{C} \in \widehat{\mathscr{C}}$ is a fuzzy $\mathcal{N}$-irreducible element of $\widehat{\mathscr{C}}$, or else $\widehat{\mathscr{C}}$ is called a fuzzy $\mathcal{N}$-reducible β-FC.*

Definition 8. *(Fuzzy $\mathcal{N}$-reduct) For a given β-FC $\widehat{\mathscr{C}}$ of U and $\widehat{\mathscr{C}'} \subseteq \widehat{\mathscr{C}}$, we call $\widehat{\mathscr{C}'}$ a fuzzy $\mathcal{N}$-reduct of $\widehat{\mathscr{C}}$ if $\Theta^{\beta}(\widehat{\mathscr{C}}) = \Theta^{\beta}(\widehat{\mathscr{C}'})$ and $\widehat{\mathscr{C}'}$ is fuzzy $\mathcal{N}$-irreducible. The collection of all fuzzy $\mathcal{N}$-reducts is denoted by $FNRed(\widehat{\mathscr{C}}) = \{\widehat{\mathscr{C}'} : \widehat{\mathscr{C}'}$ is a fuzzy $\mathcal{N}$-reduct of $\widehat{\mathscr{C}}\}$.*

Definition 8 implies that the fuzzy $\mathcal{N}$-reduct of a β-FC is the minimal subset that keeps the fuzzy β-neighborhood family invariant. Since the lower and upper approximations are determined by both the fuzzy β-neighborhood family and approximation operations, once the β-FC approximation operations are selected, the lower and upper approximations for every $X \in \mathcal{F}(U)$ only depend on the fuzzy β-neighborhood family. As a consequence, the fuzzy $\mathcal{N}$-reduction is to find the minimal subset that keeps the upper and lower approximations invariant.

Based on the above discussion, proper sufficient and necessary conditions for two β-FCs that generate the same β-FC lower and upper approximations can be obtained based on all the four types of β-FC models as defined in Definition 2.

Proposition 5. *Given two different β-FCs $\widehat{\mathscr{C}_1}$ and $\widehat{\mathscr{C}_2}$ of U, $\widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}$ generate the same primal, first, second or third type of β-FC upper and lower approximations of any $X \in \mathcal{F}(U)$ if and only if $\Theta^{\beta}(\widehat{\mathscr{C}_1}) = \Theta^{\beta}(\widehat{\mathscr{C}_2})$.*

Since $\overline{\Theta}^{\beta}(\widehat{\mathscr{C}})$ changes if and only if $\Theta^{\beta}(\widehat{\mathscr{C}})$ changes, we can also have some other sufficient and necessary conditions displayed as below:

Proposition 6. *Given two different β-FCs $\widehat{\mathscr{C}_1}$ and $\widehat{\mathscr{C}_2}$ of U, $\widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}$ generate the same primal, first, second or third type of β-FC upper and lower approximations of any $X \in \mathcal{F}(U)$ if and only if $\overline{\Theta}^{\beta}(\widehat{\mathscr{C}_1}) = \overline{\Theta}^{\beta}(\widehat{\mathscr{C}_2})$.*

According to Definitions 1 and 3, it can be known that deleting some fuzzy union reducible elements from a β-FC does not change the fuzzy β-neighborhood family, suggesting that a fuzzy union reducible element is a fuzzy $\mathcal{N}$-reducible element. However, not all fuzzy $\mathcal{N}$-reducible elements are fuzzy union reducible, as shown in the example below. Thus, the fuzzy union reducible element is a special case of the fuzzy $\mathcal{N}$-reducible element.

Example 2. *Considering the set of objects* $U = \{x_1, x_2, x_3, x_4\}$. $\widehat{\mathscr{C}} = \{\widehat{C}_1, \widehat{C}_2, \widehat{C}_3, \widehat{C}_4, \widehat{C}_5\}$ *is a* β*-FC for* $\beta = 0.5$. *Fuzzy blocks are listed as below.*

$\widehat{C}_1 = \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4}$,
$\widehat{C}_2 = \frac{0.4}{x_1} + \frac{0.8}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4}$,
$\widehat{C}_3 = \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4}$,
$\widehat{C}_4 = \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.1}{x_3} + \frac{0.3}{x_4}$,
$\widehat{C}_5 = \frac{0.7}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.4}{x_4}$.

$\Theta^\beta(\widehat{\mathscr{C}})$ *is calculated as follows:*
$\widetilde{N}^{0.5}_{x_1} = \widehat{C}_1 \bigcap \widehat{C}_3 \bigcap \widehat{C}_4 \bigcap \widehat{C}_5 = \widehat{C}_3 \bigcap \widehat{C}_4$,
$\widetilde{N}^{0.5}_{x_2} = \widehat{C}_1 \bigcap \widehat{C}_2 \bigcap \widehat{C}_4 = \widehat{C}_2 \bigcap \widehat{C}_4$,
$\widetilde{N}^{0.5}_{x_3} = \widehat{C}_3 \bigcap \widehat{C}_5 = \widehat{C}_3$,
$\widetilde{N}^{0.5}_{x_4} = \widehat{C}_1 \bigcap \widehat{C}_2 = \widehat{C}_2$.

It is obvious that $\widehat{C}_1 = \widehat{C}_2 \bigcup \widehat{C}_4$ *is both fuzzy union reducible and fuzzy* $\mathcal{N}$*-reducible; while* $\widehat{C}_5$ *is just a fuzzy* $\mathcal{N}$*-reducible element.*

Proposition 7. *For a given* β*-FC* $\widehat{\mathscr{C}}$ *of* U,
(1) $\widehat{C} \in \widehat{\mathscr{C}}$ *is fuzzy* $\mathcal{N}$*-reducible if it is fuzzy union reducible;*
(2) $\widehat{C} \in \widehat{\mathscr{C}}$ *is fuzzy union irreducible if it is fuzzy* $\mathcal{N}$*-irreducible;*
(3) $\widehat{\mathscr{C}}$ *is a fuzzy union irreducible* β*-FC if it is a fuzzy* $\mathcal{N}$*-irreducible* β*-FC.*

This proposition indicates that the fuzzy $\mathcal{N}$-reduction is a further reduction compared to the fuzzy union reduction, and the fuzzy $\mathcal{N}$-reducible element is an extension of the fuzzy union reducible element. With regard to the four β-FC models in this paper, the fuzzy union reduction is insufficient. In other words, for a fuzzy $\mathcal{N}$-reduction procedure, it is not enough to simply delete fuzzy union reducible elements from a β-FC.

Example 3. *Considering the set of objects* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $\widehat{\mathscr{C}} = \{\widehat{C}_1, \widehat{C}_2, \widehat{C}_3, \widehat{C}_4, \widehat{C}_5\}$ *is a* β*-FC for* $\beta = 0.5$. *Fuzzy blocks are listed below.*

$\widehat{C}_1 = \frac{0.8}{x_1} + \frac{0.4}{x_2} + \frac{0.2}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5}$,
$\widehat{C}_2 = \frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.6}{x_4} + \frac{0.1}{x_5}$,
$\widehat{C}_3 = \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.7}{x_5}$,
$\widehat{C}_4 = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.2}{x_3} + \frac{0.3}{x_4} + \frac{0.3}{x_5}$,
$\widehat{C}_5 = \frac{0.7}{x_1} + \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5}$.

$\Theta^\beta(\widehat{\mathscr{C}})$ *is calculated as follows:*
$\widetilde{N}^{0.5}_{x_1} = \widehat{C}_1 \bigcap \widehat{C}_3 \bigcap \widehat{C}_4 \bigcap \widehat{C}_5 = \widehat{C}_1 \bigcap \widehat{C}_3 = \widehat{C}_3 \bigcap \widehat{C}_4$, $\widetilde{N}^{0.5}_{x_2} = \widehat{C}_2$, $\widetilde{N}^{0.5}_{x_3} = \widehat{C}_3$, $\widetilde{N}^{0.5}_{x_4} = \widehat{C}_2$, $\widetilde{N}^{0.5}_{x_5} = \widehat{C}_3$.

Since $\Theta^{0.5}(\widehat{\mathscr{C}} - \{\widehat{C}_2\}) \neq \Theta^{0.5}(\widehat{\mathscr{C}})$, $\Theta^{0.5}(\widehat{\mathscr{C}} - \{\widehat{C}_3\}) \neq \Theta^{0.5}(\widehat{\mathscr{C}})$, $\Theta^{0.5}(\widehat{\mathscr{C}} - \{\widehat{C}_1\}) = \Theta^{0.5}(\widehat{\mathscr{C}} - \{\widehat{C}_4\}) = \Theta^{0.5}(\widehat{\mathscr{C}} - \{\widehat{C}_5\}) = \Theta^{0.5}(\widehat{\mathscr{C}})$, *and* $\Theta^{0.5}(\widehat{\mathscr{C}} - \{\widehat{C}_1\} - \{\widehat{C}_4\}) \neq \Theta^{0.5}(\widehat{\mathscr{C}})$, *we have* $FNR(\widehat{\mathscr{C}}) = \{\widehat{C}_1, \widehat{C}_4, \widehat{C}_5\}$, $FNI(\widehat{\mathscr{C}}) = \{\widehat{C}_2, \widehat{C}_3\}$, $FNRed(\widehat{\mathscr{C}}) = \{\{\widehat{C}_1, \widehat{C}_2, \widehat{C}_3\}, \{\widehat{C}_2, \widehat{C}_3, \widehat{C}_4\}\}$.

This example shows that deleting a fuzzy $\mathcal{N}$-reducible element may transform other original fuzzy $\mathcal{N}$-reducible elements into fuzzy $\mathcal{N}$-irreducible elements, and a β-FC may have multiple fuzzy $\mathcal{N}$-reducts.

Proposition 8. *For a given* β*-FC* $\widehat{\mathscr{C}}$ *of* U, $\widehat{C} \in \widehat{\mathscr{C}}$ *is a fuzzy* $\mathcal{N}$*-reducible element and* $\widehat{C}_1 \in \widehat{\mathscr{C}} - \{\widehat{C}\}$, *then we have*
(1) $\widehat{C}_1$ *is fuzzy* $\mathcal{N}$*-reducible in* $\widehat{\mathscr{C}}$ *if it is fuzzy* $\mathcal{N}$*-reducible in* $\widehat{\mathscr{C}} - \{\widehat{C}\}$*;*
(2) $\widehat{C}_1$ *is fuzzy* $\mathcal{N}$*-irreducible in* $\widehat{\mathscr{C}} - \{\widehat{C}\}$ *if it is fuzzy* $\mathcal{N}$*-irreducible in* $\widehat{\mathscr{C}}$.

Proof. (1) Suppose $\widehat{C}_1$ is a fuzzy $\mathcal{N}$-reducible element in $\widehat{\mathscr{C}} - \{\widehat{C}\}$, then $\Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}\}) = \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}, \widehat{C}_1\})$. Since $\widehat{C} \in \widehat{\mathscr{C}}$ is fuzzy $\mathcal{N}$-reducible, there is $\Theta^\beta(\widehat{\mathscr{C}}) = \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}\})$. Thus we have $\Theta^\beta(\widehat{\mathscr{C}}) = \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}, \widehat{C}_1\})$, it is evident that $\Theta^\beta(\widehat{\mathscr{C}}) = \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}_1\})$. That means $\widehat{C}_1$ is fuzzy $\mathcal{N}$-reducible in $\widehat{\mathscr{C}}$.

(2) Suppose $\widehat{C}_1$ is fuzzy $\mathcal{N}$-irreducible in $\widehat{\mathscr{C}}$, then $\Theta^\beta(\widehat{\mathscr{C}}) \neq \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}_1\})$, and it is evident that $\Theta^\beta(\widehat{\mathscr{C}}) \neq \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}, \widehat{C}_1\})$. Since $\widehat{C}$ is fuzzy $\mathcal{N}$-reducible in $\widehat{\mathscr{C}}$, we have $\Theta^\beta(\widehat{\mathscr{C}}) = \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}\})$. Therefore $\Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}\}) \neq \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}, \widehat{C}_1\})$, which means $\widehat{C}_1$ is fuzzy $\mathcal{N}$-irreducible in $\widehat{\mathscr{C}} - \{\widehat{C}\}$. □

As can be seen from the above proposition, deleting a fuzzy $\mathcal{N}$-reducible element in a β-FC will not transform any fuzzy $\mathcal{N}$-irreducible elements into fuzzy $\mathcal{N}$-reducible elements. As shown in Example 3, it is obvious that deleting a fuzzy $\mathcal{N}$-reducible element may cause other original fuzzy $\mathcal{N}$-reducible elements to be fuzzy $\mathcal{N}$-irreducible. Thus, the reverse does not hold.

Proposition 9. *For a given* β*-FC* $\widehat{\mathscr{C}}$ *of* U, $FNI(\widehat{\mathscr{C}}) = \bigcap FNRed(\widehat{\mathscr{C}})$.

Proof. From Proposition 8, we get that $FNI(\widehat{\mathscr{C}}) \subseteq \bigcap FNRed(\widehat{\mathscr{C}})$.

For any $\widehat{C} \in \bigcap FNRed(\widehat{\mathscr{C}})$, suppose $\widehat{C} \notin FNI(\widehat{\mathscr{C}})$, then $\Theta^\beta(\widehat{\mathscr{C}}) = \Theta^\beta(\widehat{\mathscr{C}} - \{\widehat{C}\})$. Thus there is a fuzzy $\mathcal{N}$-reduct $\widehat{\mathscr{C}'}$ which satisfies $\widehat{\mathscr{C}'} \in \widehat{\mathscr{C}} - \{\widehat{C}\}$. That means $\widehat{C} \notin \widehat{\mathscr{C}'} \in FNRed(\widehat{\mathscr{C}})$, therefore $\widehat{C} \notin \bigcap FNRed(\widehat{\mathscr{C}})$, which is a contradiction. Consequently, $FNI(\widehat{\mathscr{C}}) = \bigcap FNRed(\widehat{\mathscr{C}})$. □

As can be easily seen from the properties above, an element will not be deleted in any fuzzy $\mathcal{N}$-reduction procedure if and only if it is fuzzy $\mathcal{N}$-irreducible. In other words, all fuzzy $\mathcal{N}$-irreducible elements will remain in any fuzzy $\mathcal{N}$-reduction procedure. Therefore, the fuzzy $\mathcal{N}$-reduction procedure of a β-FC can be realized by keeping fuzzy $\mathcal{N}$-irreducible elements as the first step. Since fuzzy $\mathcal{N}$-irreducible elements are the most essential elements in fuzzy $\mathcal{N}$-reduction, it is very important to study the properties of fuzzy $\mathcal{N}$-irreducible elements.

Definition 9. *(Related set) For a given* β*-FC* $\widehat{\mathscr{C}}$ *of* U, $\widetilde{\mathbf{C}}^\beta_x = \{\widehat{C} \in \widehat{\mathscr{C}} : \widehat{C}(x) \geq \beta\}$ *is called the related set of* x.

It is evident that $\widetilde{N}^\beta_x = \cap \widetilde{\mathbf{C}}^\beta_x$.

Proposition 10. *For a given* β*-FC* $\widehat{\mathscr{C}}$ *of* U, *the following statements are equivalent:*
(1) $\widehat{C} \in FNI(\widehat{\mathscr{C}})$*;*
(2) $\widehat{C} \in \bigcap FNRed(\widehat{\mathscr{C}})$*;*
(3) There exist $x_i, x_j \in U$ *such that* $\widehat{C}$ *is the only element which satisfies* $\widehat{C}(x_i) \geq \beta$ *and* $\widehat{C}(x_j) = \widetilde{N}^\beta_{x_i}(x_j)$*;*

(4) There exist $x_i, x_j \in U$ such that $\widehat{C}(x_j) = min\{\widehat{D}(x_j) : \widehat{D} \in \widetilde{\mathbf{C}}_{x_i}^{\beta}\}$.

Proof. It is easy to see that (1)$\Longleftrightarrow$(2) and (3)$\Longleftrightarrow$(4). Next we prove (1)$\Longleftrightarrow$(3).

(1)$\Longrightarrow$(3) Let $\widehat{C} \in FNI(\widehat{\mathscr{C}})$, then there exist $x_i, x_j \in U$ such that $r_{ij} = \{\widehat{C}\}$. That means $\widehat{C}$ is the only element such that $\widehat{C}(x_i) \geq \beta$ and $\widehat{C}(x_j) = \widetilde{N}_{x_i}^{\beta}(x_j)$.

(1)$\Longleftarrow$(3) Suppose there exist $x_i, x_j \in U$ such that $\widehat{C}$ is the only element which satisfies $\widehat{C}(x_i) \geq \beta$ and $\widehat{C}(x_j) = \widetilde{N}_{x_i}^{\beta}(x_j)$, then $r_{ij} = \{\widehat{C}\}$, thus $\widehat{C} \in FNI(\widehat{\mathscr{C}})$. $\square$

From Proposition 4, Proposition 7 and Example 2, it can be known that both fuzzy union reducible elements and null elements are fuzzy $\mathcal{N}$-reducible elements, whereas fuzzy $\mathcal{N}$-reducible elements may neither be fuzzy union reducible nor null. In the following discussion, we continue to analyze which subsets of a β-FC $\widehat{\mathscr{C}}$ should be deleted and which should be reserved.

Definition 10. *(Fuzzy independent subset) For a given β-FC $\widehat{\mathscr{C}}$ of U, $\widehat{\mathscr{B}} \subseteq FNR(\widehat{\mathscr{C}})$, $\widehat{\mathscr{B}}$ is called a fuzzy independent subset of $\widehat{\mathscr{C}}$ about the fuzzy β-neighborhood family if $\Theta^{\beta}(\widehat{\mathscr{C}}) = \Theta^{\beta}(\widehat{\mathscr{C}} - \widehat{\mathscr{B}})$.*

It is evident that $FUR(\widehat{\mathscr{C}})$ is a fuzzy independent subset of $\widehat{\mathscr{C}}$ about the fuzzy β-neighborhood family.

According to the above definition, it is easy to conclude that deleting a fuzzy independent subset will not change the fuzzy β-neighborhood family. Thus, the β-FC lower and upper approximations will remain unchanged.

Definition 11. *(Fuzzy maximal independent subset) For a given β-FC $\widehat{\mathscr{C}}$ of U, $\widehat{\mathscr{B}} \subseteq FNR(\widehat{\mathscr{C}})$ is a fuzzy independent subset of $\widehat{\mathscr{C}}$, $\widehat{\mathscr{B}}$ is called a fuzzy maximal independent subset of $\widehat{\mathscr{C}}$ about the fuzzy β-neighborhood family if $\Theta^{\beta}(\widehat{\mathscr{C}}) \neq \Theta^{\beta}(\widehat{\mathscr{C}} - \widehat{\mathscr{B}} - \{\widehat{C}\})$ for each $\widehat{C} \in \widehat{\mathscr{C}} - \widehat{\mathscr{B}}$.*

It is obvious that for each $\widehat{\mathscr{C}}' \subseteq \widehat{\mathscr{C}}$, $\widehat{\mathscr{C}}' \in FNRed(\widehat{\mathscr{C}})$ if and only if there exist a fuzzy maximal independent subset $\widehat{\mathscr{B}}$ which satisfies $\widehat{\mathscr{C}}' = \widehat{\mathscr{C}} - \widehat{\mathscr{B}}$. Thus, we can obtain a fuzzy $\mathcal{N}$-reduct of a β-FC $\widehat{\mathscr{C}}$ if there is a fuzzy maximal independent subset of $\widehat{\mathscr{C}}$.

Since $FUR(\widehat{\mathscr{C}})$ is a fuzzy independent subset of $\widehat{\mathscr{C}}$ about the fuzzy β-neighborhood family, $FUR(\widehat{\mathscr{C}})$ can be deleted. However, as $FUR(\widehat{\mathscr{C}})$ is not a fuzzy maximal independent subset of $\widehat{\mathscr{C}}$, this reduction is insufficient, which explains why the reductions in [29] are inadequate.

Proposition 11. *For a given β-FC $\widehat{\mathscr{C}}$ of U, $\widehat{\mathscr{B}}$ is a fuzzy independent subset of $\widehat{\mathscr{C}}$ about the fuzzy β-neighborhood family. Then for each $\widehat{\mathscr{B}}' \subseteq \widehat{\mathscr{B}}$, $\widehat{\mathscr{B}}'$ is a fuzzy independent subset of $\widehat{\mathscr{C}}$ as well.*

Proof. Let $\Theta^{\beta}(\widehat{\mathscr{C}}) = \{(\widetilde{N}_x^{\beta})_1 : x \in U\}$, $\Theta^{\beta}(\widehat{\mathscr{C}} - \widehat{\mathscr{B}}') = \{(\widetilde{N}_x^{\beta})_2 : x \in U\}$, $\Theta^{\beta}(\widehat{\mathscr{C}} - \widehat{\mathscr{B}}) = \{(\widetilde{N}_x^{\beta})_3 : x \in U\}$. As $\widehat{\mathscr{B}}' \subseteq \widehat{\mathscr{B}} \subseteq \widehat{\mathscr{C}}$, $(\widetilde{N}_x^{\beta})_1(x) \subseteq (\widetilde{N}_x^{\beta})_2(x) \subseteq (\widetilde{N}_x^{\beta})_3(x)$ for each $x \in U$. Given that $\Theta^{\beta}(\widehat{\mathscr{C}}) = \Theta^{\beta}(\widehat{\mathscr{C}} - \widehat{\mathscr{B}})$, there is $(\widetilde{N}_x^{\beta})_1(x) = (\widetilde{N}_x^{\beta})_3(x)$. It is clear that $(\widetilde{N}_x^{\beta})_1(x) = (\widetilde{N}_x^{\beta})_2(x)$ for each $x \in U$, therefore $\Theta^{\beta}(\widehat{\mathscr{C}}) = \Theta^{\beta}(\widehat{\mathscr{C}} - \widehat{\mathscr{B}}')$. Thus, $\widehat{\mathscr{B}}'$ is a fuzzy independent subset of $\widehat{\mathscr{C}}$ about the fuzzy β-neighborhood family. $\square$

Assuming $\widehat{\mathscr{B}} \subseteq \widehat{\mathscr{C}}$, $\widehat{\mathscr{B}}$ may be not a fuzzy independent subset of $\widehat{\mathscr{C}}$ despite each $\widehat{\mathscr{B}}' \subseteq \widehat{\mathscr{B}}$ is a fuzzy independent subset of $\widehat{\mathscr{C}}$. As shown in Example 3, $\Theta^{\beta}(\widehat{\mathscr{C}} - \{\widehat{C}_1\}) = \Theta^{\beta}(\widehat{\mathscr{C}} - \{\widehat{C}_4\}) = \Theta^{\beta}(\widehat{\mathscr{C}})$, while $\Theta^{\beta}(\widehat{\mathscr{C}} - \{\widehat{C}_1\} - \{\widehat{C}_4\}) \neq \Theta^{\beta}(\widehat{\mathscr{C}})$.

IV. Granular Matrix

To keep the fuzzy β-neighborhood family $\Theta^{\beta}(\widehat{\mathscr{C}})$ invariant, it is necessary to ensure that each fuzzy β-neighborhood $\widetilde{N}_x^{\beta}$ remains unchanged. Therefore, based on the theoretical discussion in Section III, a granular matrix method is introduced.

Definition 12. *(Granular matrix) For a given β-FC $\widehat{\mathscr{C}}$ of U, $\Theta^{\beta}(\widehat{\mathscr{C}}) = \{\widetilde{N}_x^{\beta} : x \in U\}$ is the fuzzy β-neighborhood family. $R^{\beta}(\widehat{\mathscr{C}}) = (r_{ij})_{n \times n}$ is called the granular matrix of $\widehat{\mathscr{C}}$, where $r_{ij} = \{\widehat{C} \in \widetilde{\mathbf{C}}_{x_i}^{\beta} | \widehat{C}(x_j) = \widetilde{N}_{x_i}^{\beta}(x_j)\}$.*

Proposition 12. *If $\widehat{C} \in r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$, then*

(1) $\widehat{C}(x_i) \geq \beta$;

(2) For any $\widehat{C}' \in \widehat{\mathscr{C}}$, if $\widehat{C}'(x_i) \geq \beta$, then $\widehat{C}'(x_j) \geq \widehat{C}(x_j)$.

Proof. From Definition 9 and Definition 12, $r_{ij} = \{\widehat{C} \in \widetilde{\mathbf{C}}_{x_i}^{\beta} | \widehat{C}(x_j) = \widetilde{N}_{x_i}^{\beta}(x_j)\}$, $\widetilde{\mathbf{C}}_{x_i}^{\beta} = \{\widehat{C} \in \widehat{\mathscr{C}} : \widehat{C}(x_i) \geq \beta\}$.

(1) If $\widehat{C} \in r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$, then $\widehat{C} \in \widetilde{\mathbf{C}}_{x_i}^{\beta}$, it is obvious that $\widehat{C}(x_i) \geq \beta$.

(2) Since $\widehat{C} \in r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$, $\widehat{C}(x_j) = \widetilde{N}_{x_i}^{\beta}(x_j)$. For each $\widehat{C}' \in \widehat{\mathscr{C}}$, if $\widehat{C}'(x_i) \geq \beta$, then $\widehat{C}'(x_j) \geq \widetilde{N}_{x_i}^{\beta}(x_j)$, thus $\widehat{C}'(x_j) \geq \widehat{C}(x_j)$. $\square$

Proposition 13. *For a given β-FC $\widehat{\mathscr{C}}$ of U, $R^{\beta}(\widehat{\mathscr{C}}) = (r_{ij})_{n \times n}$ is the granular matrix of $\widehat{\mathscr{C}}$, then*

(1) $\widehat{\mathscr{P}} \subseteq \widehat{\mathscr{C}}$ is a fuzzy $\mathcal{N}$-reduct of $\widehat{\mathscr{C}}$ if and only if $\widehat{\mathscr{P}}$ is a minimal subset of $\widehat{\mathscr{C}}$ such that $\widehat{\mathscr{P}} \cap r_{ij} \neq \varnothing$ for each $1 \leq i, j \leq n$;

(2) $\widehat{C} \in FNI(\widehat{\mathscr{C}})$ if and only if there is $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$ such that $r_{ij} = \{\widehat{C}\}$.

Proof. (1) ($\Leftarrow$) If $\widehat{\mathscr{P}}$ is a minimal subset of $\widehat{\mathscr{C}}$ such that $\widehat{\mathscr{P}} \cap r_{ij} \neq \varnothing$ for each $1 \leq i, j \leq n$. Suppose $\widehat{C}_{ij} \in \widehat{\mathscr{P}} \cap r_{ij}$, then $\widehat{C}_{ij}(x_i) \geq \beta$ and $\widehat{C}_{ij}(x_j) = (\widetilde{N}_{x_i}^{\beta})(x_j)$. Then we have $\cap_{j=1}^{n} \widehat{C}_{ij} = \widetilde{N}_{x_i}^{\beta}(i = 1...n)$. Thus $(\widetilde{N}_x^{\beta})_{\widehat{\mathscr{C}}} = (\widetilde{N}_x^{\beta})_{\widehat{\mathscr{P}}}$ for any $x \in U$. Since $\widehat{\mathscr{P}}$ is a minimal subset, for any proper subset $\widehat{\mathscr{P}'}$ there must be r_{ij} such that $\widehat{\mathscr{P}'} \cap r_{ij} = \varnothing$. Thus for any $\widehat{C} \in \widehat{\mathscr{P}'}$, $\widehat{C}(x_j) > (\widetilde{N}_{x_i}^{\beta})_{\widehat{\mathscr{C}}}(x_j)$, then $\cap \widehat{\mathscr{P}'}(x_j) > (\widetilde{N}_{x_i}^{\beta})_{\widehat{\mathscr{C}}}(x_j)$. In other words, $(\widetilde{N}_{x_i}^{\beta})_{\widehat{\mathscr{C}}} \neq (\widetilde{N}_{x_i}^{\beta})_{\widehat{\mathscr{P}'}}$. Consequently, $\widehat{\mathscr{P}}$ is a fuzzy $\mathcal{N}$-reduct of $\widehat{\mathscr{C}}$.

($\Rightarrow$) If $\widehat{\mathscr{P}} \subseteq \widehat{\mathscr{C}}$ is a fuzzy $\mathcal{N}$-reduct of $\widehat{\mathscr{C}}$, then $(\widetilde{N}_x^{\beta})_{\widehat{\mathscr{P}}} = (\widetilde{N}_x^{\beta})_{\widehat{\mathscr{C}}}$ for any $x \in U$. Suppose $\widehat{\mathscr{P}} \cap r_{ij} = \varnothing$, then there must exist $\widehat{C}_{ij} \notin \widehat{\mathscr{P}}$ such that $\widehat{C}_{ij} \in r_{ij}$. From Definition 12, $\forall \widehat{C} \in \widehat{\mathscr{P}}, \widehat{C}(x_j) > \widehat{C}_{ij}(x_j) = \widetilde{N}_{x_i}^{\beta}(x_j)$. That is, $(\widetilde{N}_x^{\beta})_{\widehat{\mathscr{P}}} \neq (\widetilde{N}_x^{\beta})_{\widehat{\mathscr{C}}}$, which is a contradiction. Thus, $\widehat{\mathscr{P}} \cap r_{ij} \neq \varnothing$ for any $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$. Since $\widehat{\mathscr{P}}$ is a minimal subset of $\widehat{\mathscr{C}}$ such that $(\widetilde{N}_x^{\beta})_{\widehat{\mathscr{P}}} = (\widetilde{N}_x^{\beta})_{\widehat{\mathscr{C}}}$ for any $x \in U$, it is not difficult to know

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that $\widehat{\mathscr{P}}$ is a minimal subset of $\widehat{\mathscr{C}}$ such that $\widehat{\mathscr{P}} \cap r_{ij} \neq \varnothing$ for each $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$.

(2) ($\Leftarrow$) If there is $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$ such that $r_{ij} = \{\widehat{C}_0\}$. Suppose $\widehat{C}_0 \in FNR(\widehat{\mathscr{C}})$, $\widehat{\mathscr{C}}' = \widehat{\mathscr{C}} - \widehat{C}_0$, then $\forall \widehat{C} \in \widehat{\mathscr{C}}'$, $\widehat{C}(x_j) > \widehat{C}_0(x_j) = \widetilde{N}^{\beta}_{x_i}(x_j)$, that is $(\widetilde{N}^{\beta}_{x})_{\widehat{\mathscr{C}}'} \neq (\widetilde{N}^{\beta}_{x})_{\widehat{\mathscr{C}}}$, $\widehat{C}_0 \notin FNR(\widehat{\mathscr{C}})$, which is a contradiction. Thus, $\widehat{C}_0 \in FNI(\widehat{\mathscr{C}})$.

($\Rightarrow$) Suppose $\widehat{C} \in FNI(\widehat{\mathscr{C}})$, then there must exist $1 \leq i,j \leq n$ such that if we delete $\widehat{C}$ from $\widehat{\mathscr{C}}$, $\widetilde{N}^{\beta}_{x_i}(x_j)$ will be changed. Suppose $\widehat{C}' \in r_{ij}$ and $\widehat{C}' \neq \widehat{C}$, then $\widehat{C}'(x_j) = (\widetilde{N}^{\beta}_{x_i})_{\widehat{\mathscr{C}}}(x_j)$. Since $\widehat{C}' \in \widehat{\mathscr{C}} - \{\widehat{C}\}$, $(\widetilde{N}^{\beta}_{x_i})_{\widehat{\mathscr{C}}-\{\widehat{C}\}}(x_j) \leq \widehat{C}'(x_j)$, then $(\widetilde{N}^{\beta}_{x_i})_{\widehat{\mathscr{C}}}(x_j) = \widehat{C}'(x_j) = (\widetilde{N}^{\beta}_{x_i})_{\widehat{\mathscr{C}}-\{\widehat{C}\}}(x_j)$. That means deleting $\widehat{C}$ from $\widehat{\mathscr{C}}$ doesn't make $\widetilde{N}^{\beta}_{x_i}(x_j)$ change, which is a contradiction. Consequently, $r_{ij} = \{\widehat{C}\}$. $\square$

Proposition 14. *$\beta_1, \beta_2 \in [0,1]$, $\beta_1 > \beta_2$, $\widehat{\mathscr{C}}$ is a β-FC of U for both β_1 and β_2, $R^{\beta_1}(\widehat{\mathscr{C}}) = (r^1_{ij})_{n\times n}$ and $R^{\beta_2}(\widehat{\mathscr{C}}) = (r^2_{ij})_{n\times n}$ are the granular matrices of $\widehat{\mathscr{C}}$ for β_1 and β_2, respectively. We have*

(1) $\widetilde{N}^{\beta_1}_{x_i}(x_j) \geq \widetilde{N}^{\beta_2}_{x_i}(x_j)$;

(2) If $\widetilde{N}^{\beta_2}_{x_i}(x_j) \geq \beta_1$, then $r^1_{ij} = r^2_{ij}$; otherwise, $r^1_{ij} \neq r^2_{ij}$.

Proof. (1) Since $\beta_1 > \beta_2$, $\{\widehat{C}_i \in \widehat{\mathscr{C}} : \widehat{C}_i(x) \geq \beta_1\} \subseteq \{\widehat{C}_i \in \widehat{\mathscr{C}} : \widehat{C}_i(x) \geq \beta_2\}$, then $\bigcap\{\widehat{C}_i \in \widehat{\mathscr{C}} : \widehat{C}_i(x) \geq \beta_1\} \geq \bigcap\{\widehat{C}_i \in \widehat{\mathscr{C}} : \widehat{C}_i(x) \geq \beta_2\}$, thus $\widetilde{N}^{\beta_1}_{x_i}(x_j) \geq \widetilde{N}^{\beta_2}_{x_i}(x_j)$.

(2) If $\beta_1 > \beta_2$ and $\widetilde{N}^{\beta_2}_{x_i}(x_j) \geq \beta_1$, then $min\{\widehat{C}_i(x_j) : \widehat{C}_i \in \widehat{\mathscr{C}}, \widehat{C}_i(x) \geq \beta_2\} \geq \beta_1$. Then $\{\widehat{C}_i \in \widehat{\mathscr{C}} : \widehat{C}_i(x) \geq \beta_2\} = \{\widehat{C}_i \in \widehat{\mathscr{C}} : \widehat{C}_i(x) \geq \beta_1\}$, in other words, $\widetilde{N}^{\beta_1}_{x_i}(x_j) = \widetilde{N}^{\beta_2}_{x_i}(x_j)$ and $\widetilde{\mathbf{C}}^{\beta_1}_{x_i} = \widetilde{\mathbf{C}}^{\beta_2}_{x_i}$. Since $r^1_{ij} = \{\widehat{C} \in \widetilde{\mathbf{C}}^{\beta_1}_{x_i} : \widehat{C}(x_j) = \widetilde{N}^{\beta_1}_{x_i}(x_j)\}$, $r^2_{ij} = \{\widehat{C} \in \widetilde{\mathbf{C}}^{\beta_2}_{x_i} : \widehat{C}(x_j) = \widetilde{N}^{\beta_2}_{x_i}(x_j)\}$, $r^1_{ij} = r^2_{ij}$.

If $\beta_1 > \beta_2$ and $\widetilde{N}^{\beta_2}_{x_i}(x_j) < \beta_1$, then $\widetilde{N}^{\beta_1}_{x_i}(x_j) \geq \beta_1 > \widetilde{N}^{\beta_2}_{x_i}(x_j)$, thus $r^1_{ij} \neq r^2_{ij}$. $\square$

This proposition illustrates how the granular matrix changes when the parameter β changes. Next, a method for obtaining all the reducts through Boolean operation and the granular matrix is introduced.

Definition 13. *(Reduction function) For a given β-FC $\widehat{\mathscr{C}}$ of U, where $\widehat{\mathscr{C}} = \{\widehat{C}_1, \widehat{C}_2, ..., \widehat{C}_m\}$, $U = \{x_1, x_2, ..., x_n\}$, and $R^{\beta}(\widehat{\mathscr{C}}) = (r_{ij})_{n\times n}$ is the granular matrix of $\widehat{\mathscr{C}}$. A reduction function $f_{\widehat{\mathscr{C}}}$ for $\widehat{\mathscr{C}}$ is a Boolean function of m Boolean variables $\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}$ corresponding to β-FC elements $\widehat{C}_1, \widehat{C}_2, ..., \widehat{C}_m$, respectively. We define $f_{\widehat{\mathscr{C}}}(\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}) = \wedge_{1\leq i,j\leq n}(\vee r_{ij})$, where $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$ and $\vee r_{ij}$ is the disjunction of all elements in r_{ij}.*

Proposition 15. *For a given β-FC $\widehat{\mathscr{C}}$ of U, $R^{\beta}(\widehat{\mathscr{C}}) = (r_{ij})_{n\times n}$ is the granular matrix of $\widehat{\mathscr{C}}$, $f_{\widehat{\mathscr{C}}}(\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}) = \wedge_{1\leq i,j\leq n}(\vee r_{ij})$ is the reduction function. If $g_{\widehat{\mathscr{C}}}(\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}) = \vee^{l}_{k=1}(\wedge\widehat{\mathscr{P}}_k)$(where $\widehat{\mathscr{P}}_k \subseteq \widehat{\mathscr{C}}$) is the reduced disjunctive form induced from $f_{\widehat{\mathscr{C}}}$ by applying the multiplication and absorption laws, which means each element in $\widehat{\mathscr{P}}_k$ has no duplicates. Then $FNRed(\widehat{\mathscr{C}}) = \{\widehat{\mathscr{P}}_1, ..., \widehat{\mathscr{P}}_l\}$.*

Proof. For each $k = 1, 2, ..., l$, $\wedge\widehat{\mathscr{P}}_k \leq \vee r_{ij}$ for all $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$, hence $\widehat{\mathscr{P}}_k \cap r_{ij} \neq \varnothing$. Let $\widehat{\mathscr{P}}'_k = \widehat{\mathscr{P}}_k - \{\widehat{C}\}$ for any $\widehat{C} \in \widehat{\mathscr{P}}_k$, then $g_{\widehat{\mathscr{C}}} \lneqq \vee^{k-1}_{t=1}(\wedge\widehat{\mathscr{P}}_t) \vee (\wedge\widehat{\mathscr{P}}'_k) \vee (\vee^{l}_{t=k+1}(\wedge\widehat{\mathscr{P}}_t))$. If for every $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$, we have $\widehat{\mathscr{P}}'_k \cap r_{ij} \neq \varnothing$, then $\wedge\widehat{\mathscr{P}}'_k \leq \vee r_{ij}$ for every $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$. That is, $g_{\widehat{\mathscr{C}}} \geq \vee^{k-1}_{t=1}(\wedge\widehat{\mathscr{P}}_t) \vee (\wedge\widehat{\mathscr{P}}'_k) \vee (\vee^{l}_{t=k+1}(\wedge\widehat{\mathbf{P}}_t))$, which is a contradiction. Therefore there exists $r_{ij0} \in R^{\beta}(\widehat{\mathscr{C}})$ which satisfy $\widehat{\mathscr{P}}'_k \cap r_{ij0} \neq \varnothing$. Thus $\widehat{\mathscr{P}}_k$ is a reduct of $\widehat{\mathscr{C}}$.

For each $\mathbf{X} \in FNRed(\widehat{\mathscr{C}})$, it is clear that $\mathbf{X} \cap r_{ij} \neq \varnothing$ for every $r_{ij} \in R^{\beta}(\widehat{\mathscr{C}})$, then $f_{\widehat{\mathscr{C}}} \wedge (\wedge\mathbf{X}) = (\wedge(\vee r_{ij})) \wedge (\wedge\mathbf{X}) = \wedge\mathbf{X}$, which implies $\wedge\mathbf{X} \leq f_{\widehat{\mathscr{C}}} = g_{\widehat{\mathscr{C}}}$. Assuming that for every $k = 1, 2, ..., l$, there is $\widehat{\mathscr{P}}_k - \mathbf{X} \neq \varnothing$. Then, for all k, we have $\widehat{C}_k \in \widehat{\mathscr{P}}_k - \mathbf{X}$. Since there is a Boolean function Φ such that $g_{\widehat{\mathscr{C}}} = (\vee^{l}_{k=1}\widehat{C}_k) \wedge \Phi$, it is obvious that $\wedge\mathbf{X} \leq \vee^{l}_{k=1}\widehat{C}_k$. Thus there exists $\widehat{C}_{k_0}$ such that $\wedge\mathbf{X} \leq \widehat{C}_{k_0}$, that implies $\widehat{C}_{k_0} \in \mathbf{X}$, which is a contradiction. Therefore $\widehat{\mathscr{P}}_{k_0} \subseteq \mathbf{X}$ for some k_0. As both $\mathbf{X}$ and $\widehat{\mathscr{P}}_{k_0}$ are reducts, it is apparently that $\mathbf{X} = \widehat{\mathscr{P}}_{k_0}$. As a result, $FNRed(\widehat{\mathscr{C}}) = \{\widehat{\mathscr{P}}_1, ..., \widehat{\mathscr{P}}_l\}$. $\square$

The following example is intended to present the procedure of obtaining fuzzy $\mathcal{N}$-reducts based on the granular matrix.

Example 4. *Considering the set of objects $U = \{x_1, x_2, x_3, x_4\}$, $\widehat{\mathscr{C}} = \{\widehat{C}_1, \widehat{C}_2, \widehat{C}_3, \widehat{C}_4, \widehat{C}_5, \widehat{C}_6\}$ is a β-FC for $\beta = 0.5$, fuzzy blocks are listed as below.*

$\widehat{C}_1 = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.1}{x_3} + \frac{0.5}{x_4}$,
$\widehat{C}_2 = \frac{0.8}{x_1} + \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4}$,
$\widehat{C}_3 = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.1}{x_3} + \frac{0.6}{x_4}$,
$\widehat{C}_4 = \frac{0.7}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.7}{x_4}$,
$\widehat{C}_5 = \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.6}{x_4}$,
$\widehat{C}_6 = \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$.

Then the granular matrix is calculated as what follows:

$$\begin{bmatrix} \{\widehat{C}_1, \widehat{C}_3\} & \{\widehat{C}_1, \widehat{C}_2\} & \{\widehat{C}_1, \widehat{C}_3\} & \{\widehat{C}_1, \widehat{C}_2\} \\ \{\widehat{C}_4, \widehat{C}_5\} & \{\widehat{C}_4\} & \{\widehat{C}_5\} & \{\widehat{C}_5\} \\ \{\widehat{C}_6\} & \{\widehat{C}_6\} & \{\widehat{C}_5, \widehat{C}_6\} & \{\widehat{C}_6\} \\ \{\widehat{C}_6\} & \{\widehat{C}_6\} & \{\widehat{C}_1, \widehat{C}_3\} & \{\widehat{C}_1, \widehat{C}_2, \widehat{C}_6\} \end{bmatrix}$$

$f_{\widehat{\mathscr{C}}}(\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}) = (\widehat{C}_1 \vee \widehat{C}_3) \wedge (\widehat{C}_1 \vee \widehat{C}_2) \wedge (\widehat{C}_4 \vee \widehat{C}_5) \wedge \widehat{C}_4 \wedge \widehat{C}_5 \wedge \widehat{C}_6 \wedge (\widehat{C}_5 \vee \widehat{C}_6) \wedge (\widehat{C}_1 \vee \widehat{C}_2 \vee \widehat{C}_6) = (\widehat{C}_1 \wedge \widehat{C}_4 \wedge \widehat{C}_5 \wedge \widehat{C}_6) \vee (\widehat{C}_2 \wedge \widehat{C}_3 \wedge \widehat{C}_4 \wedge \widehat{C}_5 \wedge \widehat{C}_6)$.

Thus $FNRed(\widehat{\mathscr{C}}) = \{\{\widehat{C}_1, \widehat{C}_4, \widehat{C}_5, \widehat{C}_6\}, \{\widehat{C}_2, \widehat{C}_3, \widehat{C}_4, \widehat{C}_5, \widehat{C}_6\}\}$.

As shown in Example 4, although elements of the granular matrix are fuzzy set families that appear to be complex, in the following calculation based on the granularity matrix, all fuzzy sets are only used as Boolean variables (or symbols). In other words, the following calculation is irrelevant to fuzzy.

It is worth noting that some fuzzy $\mathcal{N}$-reducts maybe more concise than others. The reduct with the minimum β-FC element number is refered to as the optimum reduct. However, finding all reducts or the optimum reduct is NP-hard. Usually, only one reduct (even it is not the optimum reduct) rather than all reducts is needed in practice. Thus, a heuristic algorithm is designed to obtain a reduct (satisfactory solution).

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Proposition 16. *[28] For a given β-FC $\widehat{\mathscr{C}}$ of U, $\widetilde{N}_x^\beta = \cap \mathbf{C}_x^\beta = \cap \widetilde{Md}_x^\beta$.*

From Proposition 16, it is known that keeping $\Delta^\beta(\widehat{\mathscr{C}}) = \{\widetilde{Md}_x^\beta : x \in U\}$ invariant can ensure that $\Theta^\beta(\widehat{\mathscr{C}})$ dose not change. As a result, the lower and upper approximations are the same as the previous ones. Thus, keeping $\Delta^\beta(\widehat{\mathscr{C}})$ invariant can also help to obtain fuzzy $\mathcal{N}$-reducts, which usually includs the optimum one. To simplify this procedure, another fuzzy granular reduction algorithm based on a new granular matrix is introduced.

Definition 14. *($\mathcal{M}$-granular matrix) For a given β-FC $\widehat{\mathscr{C}}$ of U, $\Theta^\beta(\widehat{\mathscr{C}}) = \{\widetilde{N}_x^\beta : x \in U\}$ is the fuzzy β-neighborhood family. $M^\beta(\widehat{\mathscr{C}}) = (r'_{ij})_{n\times n}$ is called the $\mathcal{M}$-granular matrix of $\widehat{\mathscr{C}}$, where $r'_{ij} = \{\widehat{C} \in \widetilde{Md}_{x_i}^\beta | \widehat{C}(x_j) = \widetilde{N}_{x_i}^\beta(x_j)\}$.*

Proposition 17. *If $\widehat{C} \in r'_{ij} \in M^\beta(\widehat{\mathscr{C}})$, then (1) $\widehat{C}(x_i) \geq \beta$; (2) For any $\widehat{C}' \in \widehat{\mathscr{C}}$, if $\widehat{C}'(x_i) \geq \beta$, then $\widehat{C}'(x_j) \geq \widehat{C}(x_j)$.*

Proof. It is evident from Proposition 12. □

Proposition 18. *Suppose $\beta_1, \beta_2 \in [0,1]$, $\beta_1 > \beta_2$, $\widehat{\mathscr{C}}$ is a β-FC of U for both β_1 and β_2, $M^{\beta_1}(\widehat{\mathscr{C}}) = (r^1_{ij})_{n\times n}$ and $M^{\beta_2}(\widehat{\mathscr{C}}) = (r^2_{ij})_{n\times n}$ are the $\mathcal{M}$-granular matrices of $\widehat{\mathscr{C}}$ for β_1 and β_2, respectively. If $\widetilde{N}_{x_i}^{\beta_2}(x_j) \geq \beta_1$, then $r^1_{ij} = r^2_{ij}$; otherwise, $r^1_{ij} \neq r^2_{ij}$.*

Proof. It is evident from Proposition 14. □

Proposition 19. *For a given β-FC $\widehat{\mathscr{C}}$ of U, $M^\beta(\widehat{\mathscr{C}}) = (r'_{ij})_{n\times n}$ is the $\mathcal{M}$-granular matrix of $\widehat{\mathscr{C}}$, then*

(1) If $\widehat{\mathscr{P}}$ is a minimal subset of $\widehat{\mathscr{C}}$ such that $\widehat{\mathscr{P}} \cap r'_{ij} \neq \varnothing$ for each $1 \leq i,j \leq n$, then $\widehat{\mathscr{P}}$ is a fuzzy $\mathcal{N}$-reduct of $\widehat{\mathscr{C}}$;

(2) If $\widehat{C} \in FNI(\widehat{\mathscr{C}})$, then there is $r'_{ij} \in M^\beta(\widehat{\mathscr{C}})$ such that $r'_{ij} = \{\widehat{C}\}$.

Proof. (1) Suppose $r'_{ij} \in M^\beta(\widehat{\mathscr{C}})$, $r_{ij} \in R^\beta(\widehat{\mathscr{C}})$ and $\widehat{\mathscr{P}}$ is a minimal subset of $\widehat{\mathscr{C}}$ which satisfy $\widehat{\mathscr{P}} \cap r'_{ij} \neq \varnothing$ for each $1 \leq i,j \leq n$. From Definitions 12 and 14, we know $r'_{ij} \subseteq r_{ij}$. Since $\widehat{\mathscr{P}} \cap r'_{ij} \neq \varnothing$, we get $\widehat{\mathscr{P}} \cap r_{ij} \neq \varnothing$. Thus $\widehat{\mathscr{P}}$ is a minimal subset of $\widehat{\mathscr{C}}$ which satisfies $\widehat{\mathscr{P}} \cap r_{ij} \neq \varnothing$ for each $1 \leq i,j \leq n$. From Proposition 13, it is easy to know $\widehat{\mathscr{P}}$ is a fuzzy $\mathcal{N}$-reduct of $\widehat{\mathscr{C}}$.

(2) Suppose $\widehat{C} \in FNI(\widehat{\mathscr{C}})$, then there is $r_{ij} \in R^\beta(\widehat{\mathscr{C}})$ such that $r_{ij} = \{\widehat{C}\}$ by Proposition 13. That means $\widehat{C}$ is the only block such that $\widehat{C}(x_i) \geq \beta$ and $\widehat{C}(x_j) = \widetilde{N}_{x_i}^\beta(x_j)$. Suppose $\widehat{C}' \subseteq \widehat{C}$ and $\widehat{C}'(x_i) \geq \beta$, then $\widehat{C}' = \widehat{C}$. In other words, $\widehat{C} \in \widetilde{Md}_{x_i}^\beta$. Thus, $\{\widehat{C}\} = r'_{ij} \in M^\beta(\widehat{\mathscr{C}})$. □

Definition 15. *($\mathcal{M}$-reduction function) For a given β-FC $\widehat{\mathscr{C}}$ of U, where $\widehat{\mathscr{C}} = \{\widehat{C}_1, \widehat{C}_2, ..., \widehat{C}_m\}$, $U = \{x_1, x_2, ..., x_n\}$, $M^\beta(\widehat{\mathscr{C}}) = (r'_{ij})_{n\times n}$ is the $\mathcal{M}$-granular matrix of $\widehat{\mathscr{C}}$. A $\mathcal{M}$-reduction function $f'_{\widehat{\mathscr{C}}}$ for $\widehat{\mathscr{C}}$ is a Boolean function of m Boolean variables $\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}$ corresponding to β-FC elements $\widehat{C}_1, \widehat{C}_2, ..., \widehat{C}_m$, respectively. We define $f'_{\widehat{\mathscr{C}}}(\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}) = \wedge_{1\leq i,j\leq n}(\vee r'_{ij})$, where $r'_{ij} \in M^\beta(\widehat{\mathscr{C}})$ and $\vee r'_{ij}$ is the disjunction of all elements in r'_{ij}.*

Proposition 20. *For a given β-FC $\widehat{\mathscr{C}}$ of U, $M^\beta(\widehat{\mathscr{C}}) = \overline{(r'_{ij})_{n\times n}}$ is the $\mathcal{M}$-granular matrix of $\widehat{\mathscr{C}}$ and $f'_{\widehat{\mathscr{C}}}(\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}) = \wedge_{1\leq i,j\leq n}(\vee r'_{ij})$ is the $\mathcal{M}$-reduction function. If $g'_{\widehat{\mathscr{C}}} = \vee_{k=1}^t(\wedge \widehat{\mathscr{P}}'_k)$(where $\widehat{\mathscr{P}}'_k \subseteq \widehat{\mathscr{C}}$) is the reduced disjunctive form induced from $f'_{\widehat{\mathscr{C}}}$ by applying the multiplication and absorption laws, which means each element in $\widehat{\mathscr{P}}'_k$ has no duplicates, then $PRed(\widehat{\mathscr{C}}) = \{\widehat{\mathscr{P}}'_1, ..., \widehat{\mathscr{P}}'_t\} \subseteq FNRed(\widehat{\mathscr{C}})$.*

Proof. Suppose $r_{ij} \in R^\beta(\widehat{\mathscr{C}})$ and $r'_{ij} \in M^\beta(\widehat{\mathscr{C}})$, then we know that $r'_{ij} \subseteq r_{ij}$. It is not difficult to be proved by Proposition 15. □

This is a simple example of obtaining fuzzy $\mathcal{N}$-reducts based on $\mathcal{M}$-granular matrix.

Example 5. *Considering the set of objects $U = \{x_1, x_2, x_3, x_4\}$. Assuming $\widehat{\mathscr{C}}$ is a β-FC for $\beta = 0.5$ in Example 4, $\Delta^{0.5}(\widehat{\mathscr{C}})$ is calculated as what follows:*

$(\widetilde{Md}_{x_1}^{0.5})_{\widehat{\mathscr{C}}} = \{\widehat{C}_1\}$, $(\widetilde{Md}_{x_2}^{0.5})_{\widehat{\mathscr{C}}} = \{\widehat{C}_4, \widehat{C}_5\}$, $(\widetilde{Md}_{x_3}^{0.5})_{\widehat{\mathscr{C}}} = \{\widehat{C}_6\}$, $(\widetilde{Md}_{x_4}^{0.5})_{\widehat{\mathscr{C}}} = \{\widehat{C}_1, \widehat{C}_6\}$.

Based on $\Delta^{0.5}(\widehat{\mathscr{C}})$, we can easily obtain the $\mathcal{M}$-granular matrix:

$$\begin{bmatrix} \{\widehat{C}_1\} & \{\widehat{C}_1\} & \{\widehat{C}_1\} & \{\widehat{C}_1\} \\ \{\widehat{C}_4, \widehat{C}_5\} & \{\widehat{C}_4\} & \{\widehat{C}_5\} & \{\widehat{C}_5\} \\ \{\widehat{C}_6\} & \{\widehat{C}_6\} & \{\widehat{C}_6\} & \{\widehat{C}_6\} \\ \{\widehat{C}_6\} & \{\widehat{C}_6\} & \{\widehat{C}_1\} & \{\widehat{C}_1, \widehat{C}_6\} \end{bmatrix}$$

By the computed $\mathcal{M}$-granular matrix, we can get that:

$f'_{\widehat{\mathscr{C}}}(\overline{\widehat{C}_1}, \overline{\widehat{C}_2}, ..., \overline{\widehat{C}_m}) = \widehat{C}_1 \wedge (\widehat{C}_4 \vee \widehat{C}_5) \wedge \widehat{C}_4 \wedge \widehat{C}_5 \wedge \widehat{C}_6 \wedge (\widehat{C}_1 \vee \widehat{C}_6) = \widehat{C}_1 \wedge \widehat{C}_4 \wedge \widehat{C}_5 \wedge \widehat{C}_6$.

Thus, $\{\widehat{C}_1, \widehat{C}_4, \widehat{C}_5, \widehat{C}_6\}$ is a fuzzy $\mathcal{N}$-reduct of $\widehat{\mathscr{C}}$.

Compared with the result of Example 4, it is faster to get fuzzy $\mathcal{N}$-reducts with the $\mathcal{M}$-granular matrix, and all the fuzzy $\mathcal{N}$-reducts can only be computed by the original granular matrix. In addition, the optimum reduct is usually obtained from the results of the $\mathcal{M}$-granular matrix, and the appropriate algorithm can be chosen as needed.

V. A HEURISTIC FUZZY GRANULAR REDUCTION ALGORITHM BASED ON GRANULAR MATRIX

Based on the above discussion, we design a heuristic algorithm to obtain a fuzzy $\mathcal{N}$-reduct of a β-FC, as shown in Algorithm 1.

Example 6. *The $\mathcal{M}$-granular matrix in Example 5 is listed below:*

$$\begin{bmatrix} \{\widehat{C}_1\} & \{\widehat{C}_1\} & \{\widehat{C}_1\} & \{\widehat{C}_1\} \\ \{\widehat{C}_4, \widehat{C}_5\} & \{\widehat{C}_4\} & \{\widehat{C}_5\} & \{\widehat{C}_5\} \\ \{\widehat{C}_6\} & \{\widehat{C}_6\} & \{\widehat{C}_6\} & \{\widehat{C}_6\} \\ \{\widehat{C}_6\} & \{\widehat{C}_6\} & \{\widehat{C}_1\} & \{\widehat{C}_1, \widehat{C}_6\} \end{bmatrix}$$

By the heuristic algorithm, the collection of all singleton sets in the $\mathcal{M}$-granular matrix is $\{\widehat{C}_1, \widehat{C}_4, \widehat{C}_5, \widehat{C}_6\}$, so $FNI = \{\widehat{C}_1, \widehat{C}_4, \widehat{C}_5, \widehat{C}_6\}$. Next we delete the elements which contain any object of FNI, then the matrix becomes empty. Thus, $\{\widehat{C}_1, \widehat{C}_4, \widehat{C}_5, \widehat{C}_6\}$ is a reduct of $\widehat{\mathscr{C}}$.

Algorithm 1 A heuristic algorithm

Require: a β-FC $\widehat{\mathscr{C}}$, where $U = \{x_1, x_2, \ldots, x_n\}$, $\widehat{\mathscr{C}} = \{C_1, C_2, \ldots, C_m\}$.
Ensure: a reduct $Reduct$.

```
set FNI ← ∅;
for i ← 1 to n do
    compute Md~^β(i);
end for
set M^β ← ∅;
for i ← 1 to n, j ← 1 to n do
    M^β(i,j) = {C : C(j) = min{C(j) : C ∈ Md~^β(i)}};
    if |M^β(i,j)| = 1 and M^β(i,j) ∩ FNI = ∅ then;
        FNI = FNI ∪ M^β(i,j);
        M^β(i,j) = ∅;
    end if
end for
set Reduct ← FNI;
while M^β ≠ ∅ do
    find C which appears the most frequently in M^β
    Reduct ← C;
    for i ← 1 to n, j ← 1 to n do
        if C ∈ M^β(i,j) then
            M^β(i,j) = ∅;
        end if
    end for
end while
return Reduct
```

VI. Experiments

In this section, an experiment based on six real data sets is carried out to test the effectiveness of the proposed heuristic algorithm. The experiment mainly focuses on the following two questions:

(1) How many redundant elements are there in the existing popular granular structure, i.e. the set of fuzzy neighborhoods?

(2) What are the effects of parameters on the proposed reduction algorithm?

In this section, the algorithms are completed by Matlab R2018a(9.4) and run on a PC with Windows10 and Intel(R) Core(TM) i5-7300HQ CPU @ 2.50GHz 2.50GHz and 8.00 GB memory.

A. Preprocessing of Selected Data Sets

Six data sets were collected from University of California, Irvine (UCI) Machine Learning Repository, with the number of samples ranging from 155 to 748. The detailed overview of the selected data sets is given in Table I . All six data sets are firstly normalized to $[0, 1]$ before experiments, and denoted as (U, A), where U represents the sample set and A the feature (attribute) set. Missing values are denoted as -1. Then the following formula is adopted to compute the fuzzy granule set (the fuzzy covering) $\{B_i^a | i = 1, 2, ..., |U|\}$ induced by the attribute a.

$$B_i^a(x_j) = \begin{cases} \frac{\varepsilon - |a(x_i) - a(x_j)|}{\varepsilon} & |a(x_i) - a(x_j)| \leqslant \varepsilon \\ 0 & |a(x_i) - a(x_j)| > \varepsilon \end{cases}$$

where ε stands for the parameter of the neighborhood radius.

TABLE I: Data description.

Datasets	*Samples*	*Features*	*Classes*
Hepatitis	155	19	2
Heart	270	13	2
Bands	365	19	2
User Knowledge Modeling	403	5	4
Forest Fire	517	7	251
Blood Transfusion Service Center	748	4	2

TABLE II: Redundancy Rate of Hepatitis

β \ ε	0.1	0.2	0.3	0.4	0.5
0.5	0.8849	0.8910	0.8971	0.9002	0.9093
0.6	0.8859	0.8913	0.8913	0.8958	0.9002
0.7	0.8849	0.8893	0.8917	0.8913	0.8971
0.8	0.8835	0.8859	0.8893	0.8920	0.8910
0.9	0.8822	0.8835	0.8849	0.8859	0.8849
1.0	0.8764	0.8764	0.8764	0.8764	0.8764

B. Experimental Setup

In this experiment, a formula $RR = \frac{\sum_{i=1}^{n} |Red_i|}{m*n}$ is used to show the Redundancy Rate (or RR for short) of the fuzzy covering system induced by a data set, where m represents the number of objects, n the number of conditional features and $|Red_i|$ the cardinality of reduced covering induced by the ith feature. Based on the context above, it is evident that there are two parameters β (β is the parameter for fuzzy β-FCs) and ε in our experiment which may have an impact on Redundancy Rate. Thus, the varying range of β is set from 0.5 to 1 and ε from 0.1 to 0.5 with a step of 0.1 to see how they affect the reduction algorithm.

C. Analysis of The Results

The Redundancy Rate of six different real data sets are summarized in TABLE II-VII, of which the changes in parameters β and ε are respectively reflected from two different dimensions. The results show that the Redundancy Rate of all selected data ranges from 67.91% to 99.63%, suggesting that the proposed heuristic algorithm can reduce a large number

TABLE III: Redundancy Rate of Heart

β \ ε	0.1	0.2	0.3	0.4	0.5
0.5	0.8946	0.8966	0.8980	0.9026	0.9123
0.6	0.8952	0.8969	0.8957	0.8989	0.9026
0.7	0.8940	0.8954	0.8966	0.8957	0.8980
0.8	0.8946	0.8952	0.8954	0.8969	0.8966
0.9	0.8937	0.8946	0.8940	0.8952	0.8946
1.0	0.8917	0.8917	0.8917	0.8917	0.8917

TABLE IV: Redundancy Rate of Bands

β \ ε	0.1	0.2	0.3	0.4	0.5
0.5	0.9070	0.9097	0.9118	0.9167	0.9226
0.6	0.9071	0.9092	0.9086	0.9118	0.9167
0.7	0.9056	0.9080	0.9080	0.9086	0.9118
0.8	0.9038	0.9070	0.9080	0.9090	0.9097
0.9	0.9022	0.9038	0.9056	0.9070	0.9070
1.0	0.8963	0.9963	0.9963	0.9963	0.9963

TABLE V: Redundancy Rate of User Knowledge Modeling

β \ ε	0.1	0.2	0.3	0.4	0.5
0.5	0.6985	0.7169	0.7174	0.7224	0.7388
0.6	0.6975	0.7065	0.7264	0.7149	0.7224
0.7	0.6955	0.7035	0.7144	0.7264	0.7174
0.8	0.6910	0.6980	0.7035	0.7070	0.7169
0.9	0.6846	0.6910	0.6955	0.6980	0.6985
1.0	0.6791	0.6791	0.6791	0.6791	0.6791

TABLE VI: Redundancy Rate of Forest Fire

β \ ε	0.1	0.2	0.3	0.4	0.5
0.5	0.6993	0.7176	0.7181	0.7231	0.7395
0.6	0.6983	0.7072	0.7270	0.7156	0.7231
0.7	0.6963	0.7042	0.7151	0.7270	0.7181
0.8	0.6918	0.6988	0.7042	0.7077	0.7176
0.9	0.6854	0.6918	0.6963	0.6988	0.6993
1.0	0.6799	0.6799	0.6799	0.6799	0.6799

TABLE VII: Redundancy Rate of Blood Transfusion Service Center

β \ ε	0.1	0.2	0.3	0.4	0.5
0.5	0.9425	0.9432	0.9432	0.9449	0.9475
0.6	0.9428	0.9432	0.9435	0.9428	0.9449
0.7	0.9425	0.9425	0.9432	0.9435	0.9432
0.8	0.9415	0.9428	0.9425	0.9432	0.9432
0.9	0.9415	0.9415	0.9425	0.9428	0.9425
1.0	0.9415	0.9415	0.9415	0.9415	0.9415

of redundant granules. At the same time, the Redundancy Rate value of a data remain unchanged despite the changing parameters. Thus, Redundancy Rate is mainly determined by the data set instead of parameters, indicating that Redundancy Rate is an objective index to evaluate the granulation method.

VII. Conclusion

Granule structure is a theoretical framework for granular intelligence. To obtain a more concise granule structure, the concept of granular matrix is proposed, which lays the foundation for more efficient machine learning algorithms. A fuzzy neighborhood reduction theory is also proposed in this paper. In addition, to obtain all the granular reducts of a fuzzy β-covering, a novel reduction algorithm is proposed for the first time based on the granular matrix. Since fuzzy covering reduction is NP-hard, a heuristic greedy algorithm is constructed to obtain a reduct. Since a proper and reduced granule structure is the foundation of granular reasoning and granular intelligence, the establishment and reduction of granule structure serve as the key to a more effective and efficient intelligence method. Based on the granular matrix, a more concise granule structure (i.e. fuzzy β-covering) can be used to design the feature selection algorithm. To this end, further studies will be carried out on this topic in the future.

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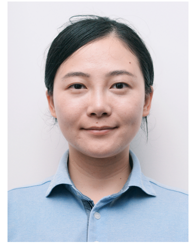

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