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Three-way cognitive concept learning via multi-granularity

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A B S T R A C T

The key strategy of the three-way decisions theory is to consider a decision-making problem as a ternary classification one (i.e. acceptance, rejection and non-commitment). Recently, this theory has been introduced into formal concept analysis for mining three-way concepts to support three-way decisions in formal contexts. That is, the three-way decisions have been performed by incorporating the idea of ternary classification into the design of extension or intension of a concept. However, the existing methods on the studies of three-way concepts are constructive, which means that the three-way concepts had been formed by defining certain concept-forming operators in advance. In order to reveal the essential characteristics of three-way concepts in making decisions from the perspective of cognition, it is necessary to reconsider three-way concepts under the framework of general concept-forming operators. In other words, axiomatic approaches are required to characterize three-way concepts. Motivated by this problem, this study mainly focuses on three-way concept learning via multi-granularity from the viewpoint of cognition. Specifically, we firstly put forward an axiomatic approach to describe three-way concepts by means of multi-granularity. Then, we design a three-way cognitive computing system to find composite three-way cognitive concepts. Furthermore, we use the idea of set approximation to simulate cognitive processes for learning three-way cognitive concepts from a given clue. Finally, numerical experiments are conducted to evaluate the performance of the proposed learning methods.

1. Introduction

Three-way decisions are one of the important ways in solving decision-making problems. Their key strategy is to consider a decision-making problem as a ternary classification one labeled by acceptance, rejection and non-commitment [60]. Up to now, substantial contributions have been made to the development of the theory of three-way decisions from various aspects. For instance, Yao [58] discussed the induction of three-way decision rules using the classical and decision-theoretic rough set models, and he also expounded the superiority of three-way decisions from the perspective of miss-classification cost [59]. Yang and Yao [53] employed the decision-theoretic rough set to model multi-agent three-way decisions. Deng and Yao [6] proposed a three-way approximation of a fuzzy set by means of the two

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parameters in the fuzzy membership function. Hu [9] established axiomatic approaches for three-way decisions and their corresponding spaces. Liang and Liu [19] built three-way decisions for the purpose of solving single-period and multi-period decision-making problems under intuitionistic fuzzy environment. Liu et al. [51] derived three-way decisions from investment decision-making problem for maximizing profit. In addition, three-way decisions have been applied to spam e-mail filtering [12], cost-sensitive face recognition [18], recommender system design [65], clustering analysis [67], and so on [68].

Cognitive computing is known as a computer system modeled on the human brain [42]. Its main purpose is to simulate human thought processes (e.g. perception, attention and learning) by computers. Cognitive learning is the function used to simulate cognitive processes such as the operations of thinking and remembering something. Generally speaking, cognitive learning can be viewed as a mathematical tool for the realization of cognitive computing. Moreover, both cognitive computing and cognitive learning have absorbed many novel methods from psychology, information theory and mathematics in the process of their development [41].

A concept, generally constituted by its extension and intension parts, is the basic unit of human cognition in philosophy [41], and is commonly used to recognize a real-world concrete entity or model a perceived-world abstract subject [42]. Up to now, many types of concepts such as abstract concepts [41], Wille's concepts [46], property-oriented concepts [7], object-oriented concepts [55,56], AFS-concepts [43] and approximate concepts [15] have been presented to meet different requirements of cognitive knowledge discovery. These well defined concepts can be distinguished from one another according to the characteristics of their intensions whose forms may be conjunctive, disjunctive or mixed. Very recently, by combining the theory of three-way decisions with formal concept analysis, Qi et al. [31,32] proposed the notion of a three-way concept to support three-way decisions in formal contexts, in which the main strategy is to incorporate the idea of ternary classification into the design of extension or intension of a concept. However, the existing methods on the studies of three-way concepts are constructive, which means that the three-way concepts were generated by introducing certain concept-forming operators in advance. In other words, researchers may define different three-way concepts with different properties, which results in a problem that which properties are the intrinsic ones of characterizing three-way concepts. The answers on this problem are important because they can help to understand the most basic decision-making mechanism of three-way concepts. So, axiomatic methods are required to look beyond appearance for the essence of three-way concepts in making decisions. The main theme of our paper is to address this problem.

Concept learning is to adopt certain approaches to learn unknown concepts from a given clue such as concept algebra system [41], queries [1], cognitive system [66], cloud model [44], set approximation [16], iteration [35], etc. According to Yao's information processing triangle [57], concept learning can be investigated from three aspects: the abstract level, brain level and machine level. More specifically, concept learning in the abstract level is to be analyzed in philosophy, mathematics and logics. For example, the formalization of the notion of a concept often refers to the principles from philosophy [14], the establishment of general concept-forming operators needs axiomatic methods [23], and logics are beneficial to the design of coherent cognitive systems. Concept learning in the brain level is to be discussed in psychology and neuroscience. For instance, the principles for perception, attention and thinking in cognitive psychology must be appropriately taken into consideration in exploring axiomatic methods [14,16]. Moreover, bi-directional recall between neurons can help to define reasonable mappings between the extension and intension parts of a concept [2]. Concept learning in the machine level is to be studied in computer science and information science. More attention has been attracted on this aspect because many kinds of effective methods [1,14,35,49,50] were developed to learn concepts from a given clue. In fact, concept learning in the abstract, brain and machine levels are relatively independent and closely related to one another. That is to say, on one hand, each of them can be researched independently. On the other hand, results from any one of them are beneficial to the better understanding of the other two. In our opinion, only by considering these three aspects in a unified framework can we have a comprehensive understanding of concept learning. The current work has an interest in the study of three-way concept learning from the abstract and machine levels.

Granular computing [63] has emerged as a unified and coherent platform of constructing, describing, and processing information granules. Currently, all kinds of models have been designed for information granules [3,4,26–30,36,45,54]. Generally speaking, the collection of information granules induced by a (resemblance, proximity, functional, etc.) relation can form a single granularity of the universe of discourse. In many practical applications, however, multiple granularities (often termed as multi-granularity) are also needed for problem-solving. For example, in a classification problem with several experts, it is a common situation that different experts have different views on dividing samples into classes. Under such a circumstance, each expert may give an independent granularity of the samples according to his or her personal preference. Then the final classification result can be obtained by effectively combining the multiple granularities from these experts. In fact, the multi-granularity view has been widely used in rough set theory. For instance, considering that multiple granularities will be generated in multi-scale datasets [47], Wu and Leung [48] studied how to select optimal granularity for optimization of the granulated information. Optimal granularity selection was also investigated from the viewpoint of local approaches [40]. Liang et al. [20] adopted the multi-granularity view to accelerate the speed of finding an approximate reduct. Based on multi-granularity, Qian et al. [33,34] put forward two novel rough set models (i.e. pessimistic and optimistic multi-granulation rough sets) for information fusion, and they designed a classifier based on these two kinds of multi-granulation rough sets. Moreover, multi-granularity has further been integrated into neighborhood-based, tolerance, covering and fuzzy rough set models [11,21,22] for complex information fusion. Also, the classical and generalized rough set models based on multi-granularity have been compared and connected with other theories such as formal concept analysis and cost-sensitive classification [17,39,52,62]. In this paper, multi-granularity will be used to discuss three-way concept learning.

The cognitive viewpoint has been commonly adopted in the study of concept learning [2,14,16,49,50,57,66] because it has a wide application background in simulating intelligence behaviors of the brain including thinking, learning and reasoning. For this reason, this idea will also be incorporated into three-way concept learning in this paper. In summary, three-way cognitive concept learning via multi-granularity deserves to be studied, which is the main issue that our current work focuses on. Specifically, we propose an axiomatic approach to describe three-way concepts based on multi-granularity, establish a three-way cognitive computing system for finding composite three-way cognitive concepts, and simulate cognitive processes for learning three-way concepts from a given clue. Besides, some numerical experiments are conducted to assess the performance of the proposed learning methods. The main contribution of this paper is to reveal the essential idea of three-way concepts in solving decision-making problems from the aspect of cognition, meaning that we will clarify which properties of three-way concepts are intrinsic.

The remainder of this paper is organized as follows. Section 2 analyzes cognitive mechanism of forming three-way concepts based on multi-granularity and three-way-decision-making principles. Moreover, the notions of three-way cognitive operators, three-way cognitive concepts and three-way granular concepts are proposed. Some important properties are also discussed. Section 3 designs a three-way cognitive computing system which is in fact a dynamic process to update three-way granular concepts. Section 4 employs the idea of set approximation to simulate cognitive processes for learning three-way cognitive concepts from a given clue. Section 5 conducts some numerical experiments to evaluate the performance of the proposed learning methods. The paper is then concluded with a brief summary and an outlook for further research.

2. Cognitive mechanism of forming three-way concepts

In this section, we analyze cognitive mechanism of forming three-way concepts based on multi-granularity and three-way-decision-making principles. Throughout the paper, we denote by U a nonempty object set, i.e., the universe of discourse, and A an attribute set.

2.1. Basic notions

We first introduce the notions of three-way decisions, three-way quotient set and its power set.

In accordance with the notations in rough set theory [25], we still call the sets inducing three-way decisions (i.e., acceptance, rejection and non-commitment [60]) as positive, negative and boundary regions [5,10,64,69]. Moreover, the positive, negative and boundary regions are not distinguished from their respective three-way decisions in the subsequent discussions.

For $x \in U$ and $A_i \subseteq A$, let $f_{A_i}(x)$ be an evaluation function associated to A_i. Then, given two parameters α and β with $\beta < \alpha$, the positive, negative and boundary regions can be formalized as follows:

(i) positive region: $X_i = \{x \in U \mid f_{A_i}(x) \geq \alpha\}$,
(ii) negative region: $Y_i = \{x \in U \mid f_{A_i}(x) \leq \beta\}$,
(iii) boundary region: $Z_i = U - X_i - Y_i$.

It should be pointed out that the evaluation function f_{A_i} can be defined according to the practical background of the problem to be solved. In addition, the assignment of values to the parameters α and β is performed by an expert based on his or her experience in the field that the problem belongs to.

Furthermore, we say that X_i, Y_i and Z_i are three-way decisions induced by A_i with the help of α and β. In fact, from granular computing, three-way decisions X_i, Y_i and Z_i form a granularity of U by eliminating empty sets. Note that these three-way decisions satisfy $X_i \cup Y_i \cup Z_i = U$. So, it is sufficient to describe three-way decisions by any two of them. Hereinafter, we choose (X_i, Y_i) to represent three-way decisions when no confusion is caused.

Let S be an index set. Suppose that (X_i, Y_i) $(i \in S)$ are a series of three-way decisions induced by multiple subsets A_i $(i \in S)$ of A. Then, for each $i \in S$, three-way decisions (X_i, Y_i) can form a granularity of U by eliminating empty sets. Therefore, (X_i, Y_i) $(i \in S)$ can be viewed as a result of multi-granularity of U.

Moreover, if the multiple subsets A_i $(i \in S)$ constitute a partition of A, then $\mathcal{Q}(A) = \{A_i \mid i \in S\}$ is called a three-way quotient set of A. For convenience, we denote the power set of $\mathcal{Q}(A)$ by $2^{\mathcal{Q}(A)}$. Here, every $B \in 2^{\mathcal{Q}(A)}$ can be considered as a group of knowledge jointly inducing three-way decisions.

2.2. Three-way cognitive operators induced by multi-granularity and three-way-decision-making principles

For two three-way decisions (X_i, Y_i) and (X_j, Y_j) of U, if $X_i \subseteq X_j$ and $Y_i \subseteq Y_j$, then (X_j, Y_j) is said to be more effective than (X_i, Y_i), which we denote by $(X_i, Y_i) \preceq (X_j, Y_j)$. Moreover, if $(X_i, Y_i) \preceq (X_j, Y_j)$, we also say that (X_i, Y_i) is decision-consistent with respect to (X_j, Y_j).

The set of three-way decisions, induced by multi-granularity of U, is denoted by $\mathcal{T}(U)$. Furthermore, the intersection and union in $\mathcal{T}(U)$ are respectively defined as

$$(X_i, Y_i) \cap (X_j, Y_j) = (X_i \cap X_j, Y_i \cap Y_j),$$
$$(X_i, Y_i) \cup (X_j, Y_j) = (X_i \cup X_j, Y_i \cup Y_j).$$

Table 1
A reviewing dataset.

Manuscript	Domain 1		Domain 2		Domain 3		
	Reviewer r_1	Reviewer r_2	Reviewer r_3	Reviewer r_4	Reviewer r_5	Reviewer r_6	Reviewer r_7
x_1	Accept	Accept					
x_2	Accept	Reject					
x_3	Reject	Accept					
x_4			Accept	Reject			
x_5			Accept	Accept			
x_6			Reject	Reject			
x_7					Accept	Reject	Accept
x_8					Accept	Accept	Accept
x_9					Reject	Reject	Accept

Now, we discuss what the mappings $\mathcal{H}: 2^{\mathcal{Q}(A)} \to \mathcal{T}(U)$ and $\mathcal{L}: \mathcal{T}(U) \to 2^{\mathcal{Q}(A)}$ need to obey when they are used to form three-way concepts?

Three-way-decision-making principle I: According to sequential or dynamic three-way decisions [9,18,61], the more knowledge we use to induce three-way decisions, the more effective the induced three-way decisions are. From this principle, we have

$$B_i \subseteq B_j \Rightarrow \mathcal{H}(B_i) \preccurlyeq \mathcal{H}(B_j). \quad (1)$$

Three-way-decision-making principle II: Three-way decisions made by the whole group are less effective than or as effective as the combination of those made by its sub-groups. From this principle, we have

$$\mathcal{H}(B_i \cup B_j) \preccurlyeq \mathcal{H}(B_i) \cup \mathcal{H}(B_j). \quad (2)$$

Three-way-decision-making principle III: Whether or not the knowledge is selected depends on how decision-consistent its induced three-way decisions are with respect to the target three-way decisions (*X*, *Y*). From this principle, we obtain

$$\mathcal{L}(X, Y) = \{A_i \in \mathcal{Q}(A) \mid \mathcal{H}(\{A_i\}) \preccurlyeq (X, Y)\}. \quad (3)$$

In what follows, Eqs. (1)–(3) are used as conditions to define three-way cognitive operators based on the mappings $\mathcal{H}$ and $\mathcal{L}$.

Definition 1. Given two mappings $\mathcal{H}: 2^{\mathcal{Q}(A)} \to \mathcal{T}(U)$ and $\mathcal{L}: \mathcal{T}(U) \to 2^{\mathcal{Q}(A)}$, if for any $B_i, B_j \in 2^{\mathcal{Q}(A)}$ and $(X, Y) \in \mathcal{T}(U)$, the following properties hold:
(i) $B_i \subseteq B_j \Rightarrow \mathcal{H}(B_i) \preccurlyeq \mathcal{H}(B_j)$,
(ii) $\mathcal{H}(B_i \cup B_j) \preccurlyeq \mathcal{H}(B_i) \cup \mathcal{H}(B_j)$,
(iii) $\mathcal{L}(X, Y) = \{A_i \in \mathcal{Q}(A) \mid \mathcal{H}(\{A_i\}) \preccurlyeq (X, Y)\}$, then $\mathcal{H}$ and $\mathcal{L}$ are called three-way cognitive operators.
Note that the reason of calling $\mathcal{H}$ and $\mathcal{L}$ as three-way cognitive operators is as follows:
(1) both $\mathcal{H}$ and $\mathcal{L}$ involve three-way decisions, i.e., the co-domain of $\mathcal{H}$ and the domain of $\mathcal{L}$;
(2) $\mathcal{H}$ and $\mathcal{L}$ can be jointly used to form concepts, i.e., recognition of concepts.
In addition, it should be pointed out that the properties (i), (ii) and (iii) used to define three-way cognitive operators are from three-way-decision-making principles I, II and III, respectively. In other words, there are explicit semantics for these properties. In fact, the properties (i)–(iii) are very important because they can be jointly used as axioms to characterize three-way concepts. So, these properties can be considered as the intrinsic ones of characterizing three-way concepts.

Remark 1. For three-way cognitive operators $\mathcal{H}$ and $\mathcal{L}$, we say that three-way decisions (*X*, *Y*) induced by a nonempty set $B \in 2^{\mathcal{Q}(A)}$ (i.e. $\mathcal{H}(B) = (X, Y)$) are trivial if *X* and *Y* are empty simultaneously. Hereinafter, three-way decisions induced by any nonempty set $B \in 2^{\mathcal{Q}(A)}$ are assumed to be not trivial.

Remark 2. For three-way decisions (X_i, Y_i) and (X_j, Y_j) induced by two nonempty sets $B_i, B_j \in 2^{\mathcal{Q}(A)}$, if $X_i \cap Y_j = \emptyset$ and $Y_i \cap X_j = \emptyset$, we say that (X_i, Y_i) and (X_j, Y_j) are uncontradictory with each other. In the rest of this paper, three-way decisions induced by any two nonempty sets $B_i, B_j \in 2^{\mathcal{Q}(A)}$ are assumed to be uncontradictory with each other.
For conciseness, we also write $\mathcal{H}(\{A_i\})$ as $\mathcal{H}(A_i)$ when no confusion is caused.

Example 1. Table 1 depicts a dataset of nine manuscripts evaluated by seven reviewers who are from three domains. That is, the first two reviewers are from Domain 1, the second two from Domain 2, and the remainder from Domain 3. In the table, null value in the cross of a row and a column means that the manuscript in this row was not assigned to be evaluated by the reviewer from this column. It is easy to observe that the manuscripts 1–3 fall into the first domain, the manuscripts 4–6 the second domain, and the manuscripts 7–9 the third domain.
Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ be the set of nine manuscripts, and $A = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$ be the set of seven reviewers. Then, $A_1 = \{r_1, r_2\}$, $A_2 = \{r_3, r_4\}$ and $A_3 = \{r_5, r_6, r_7\}$ are just the reviewers from three domains, respectively. Since

A_1, A_2 and A_3 form a partition of A, then $\mathcal{Q}(A) = \{A_1, A_2, A_3\}$ is a three-way quotient set of A. Moreover, its power set is as follows:

$$2^{\mathcal{Q}(A)} = \{\emptyset, \{A_1\}, \{A_2\}, \{A_3\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_2, A_3\}, \{A_1, A_2, A_3\}\}.$$

Take $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$. Suppose the evaluation function $f_{B_i}(x)$ $(B_i \in 2^{\mathcal{Q}(A)},\ x \in U)$ is the ratio of the number of Accepts given to x to that of reviewers assigned to x under the columns $\cup B_i$. Note that x will be put into the boundary region directly if the total number of Accepts and Rejects given to x under the columns $\cup B_i$ is less than or equal to 1. In other words, a non-commitment decision will be made to x if it does not receive enough evaluations. Then we can generate the following three-way decisions:

- positive region induced by $\{A_1\}$: $X_1 = \{x \in U \mid f_{\{A_1\}}(x) \geq \alpha\} = \{x_1\}$,
- negative region induced by $\{A_1\}$: $Y_1 = \{x \in U \mid f_{\{A_1\}}(x) \leq \beta\} = \emptyset$,
- positive region induced by $\{A_2\}$: $X_2 = \{x \in U \mid f_{\{A_2\}}(x) \geq \alpha\} = \{x_5\}$,
- negative region induced by $\{A_2\}$: $Y_2 = \{x \in U \mid f_{\{A_2\}}(x) \leq \beta\} = \{x_6\}$,
- positive region induced by $\{A_3\}$: $X_3 = \{x \in U \mid f_{\{A_3\}}(x) \geq \alpha\} = \{x_7, x_8\}$,
- negative region induced by $\{A_3\}$: $Y_3 = \{x \in U \mid f_{\{A_3\}}(x) \leq \beta\} = \{x_9\}$,
- positive region induced by $\{A_1, A_2\}$: $X_4 = \{x \in U \mid f_{\{A_1, A_2\}}(x) \geq \alpha\} = \{x_1, x_5\}$,
- negative region induced by $\{A_1, A_2\}$: $Y_4 = \{x \in U \mid f_{\{A_1, A_2\}}(x) \leq \beta\} = \{x_6\}$,
- positive region induced by $\{A_1, A_3\}$: $X_5 = \{x \in U \mid f_{\{A_1, A_3\}}(x) \geq \alpha\} = \{x_1, x_7, x_8\}$,
- negative region induced by $\{A_1, A_3\}$: $Y_5 = \{x \in U \mid f_{\{A_1, A_3\}}(x) \leq \beta\} = \{x_9\}$,
- positive region induced by $\{A_2, A_3\}$: $X_6 = \{x \in U \mid f_{\{A_2, A_3\}}(x) \geq \alpha\} = \{x_5, x_7, x_8\}$,
- negative region induced by $\{A_2, A_3\}$: $Y_6 = \{x \in U \mid f_{\{A_2, A_3\}}(x) \leq \beta\} = \{x_6, x_9\}$,
- positive region induced by $\{A_1, A_2, A_3\}$: $X_7 = \{x \in U \mid f_{\{A_1, A_2, A_3\}}(x) \geq \alpha\} = \{x_1, x_5, x_7, x_8\}$,
- negative region induced by $\{A_1, A_2, A_3\}$: $Y_7 = \{x \in U \mid f_{\{A_1, A_2, A_3\}}(x) \leq \beta\} = \{x_6, x_9\}$.

Thus, $\mathcal{T}(U) = \{(\emptyset, \emptyset), (X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4), (X_5, Y_5), (X_6, Y_6), (X_7, Y_7)\}$ is the set of three-way decisions induced by multi-granularity of U. It should be pointed out that the trivial three-way decisions $(\emptyset, \emptyset)$ is forcibly included in $\mathcal{T}(U)$ in order to establish the following mappings:

$$\begin{aligned} \mathcal{H}: \quad & \emptyset \mapsto (\emptyset, \emptyset), \{A_1\} \mapsto (X_1, Y_1), \{A_2\} \mapsto (X_2, Y_2), \{A_3\} \mapsto (X_3, Y_3), \\ & \{A_1, A_2\} \mapsto (X_4, Y_4), \{A_1, A_3\} \mapsto (X_5, Y_5), \{A_2, A_3\} \mapsto (X_6, Y_6), \{A_1, A_2, A_3\} \mapsto (X_7, Y_7) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}: \quad & (\emptyset, \emptyset) \mapsto \emptyset, && (X_1, Y_1) \mapsto \{A_1\}, && (X_2, Y_2) \mapsto \{A_2\}, && (X_3, Y_3) \mapsto \{A_3\}, \\ & (X_4, Y_4) \mapsto \{A_1, A_2\}, && (X_5, Y_5) \mapsto \{A_1, A_3\}, && (X_6, Y_6) \mapsto \{A_2, A_3\}, && (X_7, Y_7) \mapsto \{A_1, A_2, A_3\}. \end{aligned}$$

Then, based on Definition 1, the mappings $\mathcal{H}$ and $\mathcal{L}$ are three-way cognitive operators.

Proposition 1. *Let $\mathcal{H}$ and $\mathcal{L}$ be three-way cognitive operators. Then for any nonempty set $B \in 2^{\mathcal{Q}(A)}$, we have*

$$\mathcal{H}(B) = \bigcup_{A_i \in B} \mathcal{H}(A_i). \tag{4}$$

Proof. To complete the proof, it is sufficient to show $\mathcal{H}(\{A_i, A_j\}) = \mathcal{H}(A_i) \cup \mathcal{H}(A_j)$, where A_i, $A_j \in B$. By Eq. (1), we have $\mathcal{H}(A_i) \cup \mathcal{H}(A_j) \preccurlyeq \mathcal{H}(\{A_i, A_j\})$ due to $\mathcal{H}(A_i) \preccurlyeq \mathcal{H}(\{A_i, A_j\})$ and $\mathcal{H}(A_j) \preccurlyeq \mathcal{H}(\{A_i, A_j\})$. By combining $\mathcal{H}(A_i) \cup \mathcal{H}(A_j) \preccurlyeq \mathcal{H}(\{A_i, A_j\})$ with Eq. (2), we conclude $\mathcal{H}(\{A_i, A_j\}) = \mathcal{H}(A_i) \cup \mathcal{H}(A_j)$. □

Proposition 2. *Let $\mathcal{H}$ and $\mathcal{L}$ be three-way cognitive operators. For any $B \in 2^{\mathcal{Q}(A)}$ and $(X, Y), (X_i, Y_i), (X_j, Y_j) \in \mathcal{T}(U)$, we have the following properties:*

$$B \subseteq \mathcal{LH}(B); \tag{5}$$

$$\mathcal{HL}(X, Y) \preccurlyeq (X, Y); \tag{6}$$

$$(X_i, Y_i) \preccurlyeq (X_j, Y_j) \Rightarrow \mathcal{L}(X_i, Y_i) \subseteq \mathcal{L}(X_j, Y_j), \tag{7}$$

where $\mathcal{HL}(\bullet)$ and $\mathcal{LH}(\bullet)$ represent the composite mappings $\mathcal{H}(\mathcal{L}(\bullet))$ and $\mathcal{L}(\mathcal{H}(\bullet))$, respectively.

Proof. Firstly, we prove Eq. (5). For any $A_i \in B$, we have $\mathcal{H}(A_i) \preccurlyeq \mathcal{H}(B)$ according to Eq. (1). Based on Eq. (3), we obtain $A_i \in \mathcal{LH}(B)$. As a result, $B \subseteq \mathcal{LH}(B)$ is true.

Secondly, we prove Eq. (6). For any $A_i \in \mathcal{L}(X, Y)$, by Eq. (3), we get $\mathcal{H}(A_i) \preccurlyeq (X, Y)$. It can be seen from Proposition 1 that $\mathcal{HL}(X, Y) = \bigcup_{A_i \in \mathcal{L}(X,Y)} \mathcal{H}(A_i) \preccurlyeq (X, Y)$.

Finally, we prove Eq. (7). Suppose $(X_i, Y_i) \preceq (X_j, Y_j)$. Then, for any $A_i \in \mathcal{L}(X_i, Y_i)$, we know from Eq. (3) that $\mathcal{H}(A_i) \preccurlyeq (X_i, Y_i)$ is true. Moreover, we have $\mathcal{H}(A_i) \preccurlyeq (X_j, Y_j)$. Consequently, $A_i \in \mathcal{L}(X_j, Y_j)$ is proved. □

It deserves to point out that based on Eqs. (1), (5), (6) and (7), the pair $(\mathcal{H}, \mathcal{L})$ forms an isotone Galois connection [24,46] between $2^{\mathcal{Q}(A)}$ and $\mathcal{T}(U)$. This means that the mappings $\mathcal{H}$ and $\mathcal{L}$ can be jointly used to induce concepts. By the way, three-way cognitive operators $\mathcal{H}$ and $\mathcal{L}$ are completely different from the classical ones [16] which form an antitone Galois connection between 2^A and 2^U.

2.3. Three-way cognitive concepts and three-way granular concepts

In this subsection, we put forward the notion of a three-way cognitive concept and discuss information granules for three-way cognitive concepts.

Definition 2. Let $\mathcal{H}$ and $\mathcal{L}$ be three-way cognitive operators. For $B \in 2^{\mathcal{Q}(A)}$ and $(X, Y) \in \mathcal{T}(U)$, if $\mathcal{H}(B) = (X, Y)$ and $\mathcal{L}(X, Y) = B$, we say that $\langle (X, Y), B\rangle$ is a three-way concept under the cognitive operators $\mathcal{H}$ and $\mathcal{L}$ (or simply a three-way cognitive concept). In this case, (X, Y) and B are called the extent and intent of the three-way cognitive concept $\langle (X, Y), B\rangle$, respectively. Henceforth, the set of all three-way cognitive concepts is denoted by $\mathfrak{B}(2^{\mathcal{Q}(A)}, \mathcal{T}(U), \mathcal{H}, \mathcal{L})$.

Three-way cognitive concepts are different from triadic concepts [13] which consist of extent, intent and modus. The reasons are as follows:

(i) The extent of a three-way cognitive concept is constituted by positive region, negative region and boundary region, while that of a triadic concept is only a subset of the universe of discourse.

(ii) The attribute sets A_i $(i \in S)$ used to induce basic three-way decisions are pairwise disjoint (see, e.g., those under Domains 1–3 in Table 1), while the attribute sets under conditions of a triadic context are the same. That is to say, the relationship between the extent and intent of a three-way cognitive concept is different from that of a triadic concept. More specifically, the former claims that the intent is considered as an evaluation function to partition the universe of discourse into positive, negative and boundary regions for generating extent, while the latter emphasizes that each object in extent has all the attributes in intent under every condition in modus.

The infimum ($\bigwedge$) and supremum ($\bigvee$) among three-way cognitive concepts $\mathfrak{B}(2^{\mathcal{Q}(A)}, \mathcal{T}(U), \mathcal{H}, \mathcal{L})$ are respectively defined as:

$$\begin{aligned} &\langle (X_i, Y_i), B_i\rangle \bigwedge \langle (X_j, Y_j), B_j\rangle = \langle \mathcal{HL}((X_i, Y_i) \cap (X_j, Y_j)), B_i \cap B_j\rangle, \\ &\langle (X_i, Y_i), B_i\rangle \bigvee \langle (X_j, Y_j), B_j\rangle = \langle (X_i, Y_i) \cup (X_j, Y_j), \mathcal{LH}(B_i \cup B_j)\rangle. \end{aligned} \tag{8}$$

Example 2 (Continued with Example 1)**.** Based on Definition 2, we can obtain the following three-way cognitive concepts for Example 1:

$$\begin{array}{llll} \langle(\{x_1, x_5, x_7, x_8\}, \{x_6, x_9\}), \mathcal{Q}(A)\rangle, & \langle(\emptyset, \emptyset), \emptyset\rangle, & \langle(\{x_1, x_5\}, \{x_6\}), \{A_1, A_2\}\rangle, & \langle(\{x_1, x_7, x_8\}, \{x_9\}), \{A_1, A_3\}\rangle, \\ \langle(\{x_5, x_7, x_8\}, \{x_6, x_9\}), \{A_2, A_3\}\rangle, & \langle(\{x_1\}, \emptyset), \{A_1\}\rangle, & \langle(\{x_5\}, \{x_6\}), \{A_2\}\rangle, & \langle(\{x_7, x_8\}, \{x_9\}), \{A_3\}\rangle, \end{array}$$

where $A_1 = \{r_1, r_2\}$, $A_2 = \{r_3, r_4\}$ and $A_3 = \{r_5, r_6, r_7\}$.

Definition 3. Let $\mathcal{H}$ and $\mathcal{L}$ be three-way cognitive operators. Then $\mathcal{H}^G = \{\{A_i\} \to \mathcal{H}(A_i) \mid A_i \in \mathcal{Q}(A)\}$ is called information granules of $\mathcal{H}$.

According to Eq. (4), the information granules $\mathcal{H}^G$ can be used to generate the mapping $\mathcal{H}$.

Proposition 3. *Let $\mathcal{H}$ and $\mathcal{L}$ be three-way cognitive operators. Then for any $B \in 2^{\mathcal{Q}(A)}$, $\langle \mathcal{H}(B), \mathcal{LH}(B)\rangle$ is a three-way cognitive concept.*

Proof. The conclusion is immediate from Definition 2 and Proposition 2. □

Proposition 3 is further used to define the notion of a three-way granular concept by taking $B = \{A_i\}$ and the formalization is given below.

Definition 4. Let $\mathcal{H}$ and $\mathcal{L}$ be three-way cognitive operators. Then for any singleton set $\{A_i\} \in 2^{\mathcal{Q}(A)}$, we say that $\langle \mathcal{H}(A_i), \mathcal{LH}(A_i)\rangle$ is a three-way granular concept.

Proposition 4. *Let $\mathcal{H}$ and $\mathcal{L}$ be three-way cognitive operators. Then for any $\langle (X, Y), B\rangle \in \mathfrak{B}(2^{\mathcal{Q}(A)}, \mathcal{T}(U), \mathcal{H}, \mathcal{L})$, we have*

$$\langle (X, Y), B\rangle = \bigvee_{A_i \in B} \langle \mathcal{H}(A_i), \mathcal{LH}(A_i)\rangle. \tag{9}$$

Proof. The conclusion can be obtained directly from Eqs. (4) and (8). □

Proposition 4 indicates that any three-way cognitive concept can be induced by integrating three-way granular concepts. Thus, from granular computing, three-way granular concepts can be considered as the information granules of $\mathfrak{B}(2^{\mathcal{Q}(A)}, \mathcal{T}(U), \mathcal{H}, \mathcal{L})$. Hereinafter, we denote the collection of the information granules by $G_{\mathcal{HL}}$. That is,

$$G_{\mathcal{HL}} = \{\langle \mathcal{H}(A_i), \mathcal{LH}(A_i)\rangle \mid A_i \in \mathcal{Q}(A)\}. \tag{10}$$

Example 3 (Continued with Example 2)**.** Note that the following equations hold for Example 2:

$$\begin{array}{ll} \mathcal{H}(A_1) = (\{x_1\}, \emptyset), & \mathcal{LH}(A_1) = \mathcal{L}(\{x_1\}, \emptyset) = \{A_1\}, \\ \mathcal{H}(A_2) = (\{x_5\}, \{x_6\}), & \mathcal{LH}(A_2) = \mathcal{L}(\{x_5\}, \{x_6\}) = \{A_2\}, \\ \mathcal{H}(A_3) = (\{x_7, x_8\}, \{x_9\}), & \mathcal{LH}(A_3) = \mathcal{L}(\{x_7, x_8\}, \{x_9\}) = \{A_3\}, \end{array}$$

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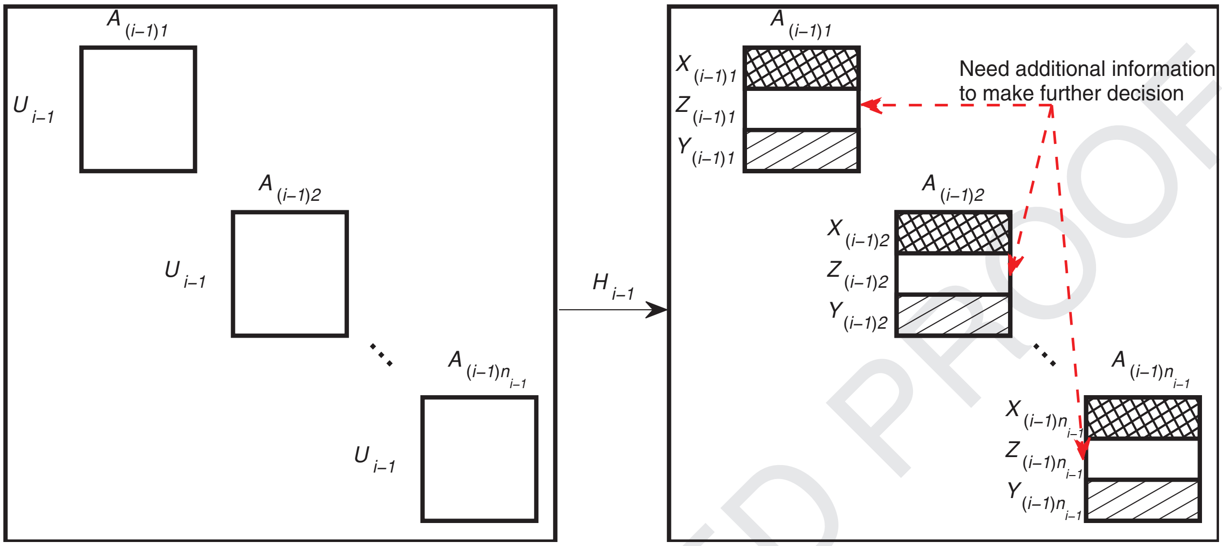

Fig. 1. The graph of $\mathcal{H}_{i-1}$.

where $A_1 = \{r_1, r_2\}$, $A_2 = \{r_3, r_4\}$ and $A_3 = \{r_5, r_6, r_7\}$. Thus, based on Definition 4, we know that $\langle(\{x_1\}, \emptyset), \{A_1\}\rangle$, $\langle(\{x_5\}, \{x_6\}), \{A_2\}\rangle$ and $\langle(\{x_7, x_8\}, \{x_9\}), \{A_3\}\rangle$ are three-way granular concepts. That is,

$$G_{\mathcal{HL}} = \{\langle(\{x_1\}, \emptyset), \{A_1\}\rangle, \langle(\{x_5\}, \{x_6\}), \{A_2\}\rangle, \langle(\{x_7, x_8\}, \{x_9\}), \{A_3\}\rangle\},$$

which can be further used to induce other three-way cognitive concepts.

3. Three-way cognitive computing system

From cognitive computing, concepts should be updated to simulate intelligence behaviors of the brain when information is updated periodically. For instance, in Example 1, nine manuscripts were evaluated by seven reviewers. As time goes by, on one hand, new manuscripts will arrive. On the other hand, those falling into the boundary regions need to be evaluated by inviting additional reviewers. In this case, it is necessary to update three-way granular concepts for supporting a further decision of the manuscripts with non-commitment decisions.

Motivated by the above problem, we propose in this section a three-way cognitive computing system to update three-way granular concepts as objects and/or attributes increase in batches. Before embarking on this issue, we introduce some notations.

To facilitate our subsequent discussion, n attribute sets $A_1, A_2, \ldots, A_n$ with $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$ are denoted by $\{A_t | t \in S\}$, where $S = \{1, 2, \ldots, n\}$. Similarly, n object sets $U_1, U_2, \ldots, U_n$ with $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n$ are denoted by $\{U_t | t \in S\}$. Let $\Delta A_{i-1} = A_i - A_{i-1}$ and $\Delta U_{i-1} = U_i - U_{i-1}$. Moreover, for any $i \in S$, we denote by $2^{\mathcal{Q}(A_i)}$ the power set of three-way quotient set of A_i, and by $\mathcal{T}(U_i)$ the set of three-way decisions of U_i.

For any $A_{i-1s} \in \mathcal{Q}(A_{i-1})$, if there exists $A_{it} \in \mathcal{Q}(A_i)$ such that $A_{i-1s} \subseteq A_{it}$, then $\mathcal{Q}(A_i)$ is called a generalization of $\mathcal{Q}(A_{i-1})$ or equivalently $\mathcal{Q}(A_{i-1})$ is a specification of $\mathcal{Q}(A_i)$, where $i-1$, s, i and t are subscript indices. We denote this generalization/specification relation by $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$. Such a relation is easy to be understood in the real world. For instance, in Example 1, new invited reviewers for evaluating the manuscripts falling into the boundary regions must be from Domain 1, Domain 2, Domain 3 or a new domain.

Suppose that

$$\mathcal{H}_{i-1} : 2^{\mathcal{Q}(A_{i-1})} \to \mathcal{T}(U_{i-1}) \tag{11}$$

is a mapping from $2^{\mathcal{Q}(A_{i-1})}$ to $\mathcal{T}(U_{i-1})$ and

$$\mathcal{H}_{\Delta U_{i-1}} : 2^{\mathcal{Q}(A_{i-1})} \to \mathcal{T}(\Delta U_{i-1}) \tag{12}$$

is a mapping from $2^{\mathcal{Q}(A_{i-1})}$ to $\mathcal{T}(\Delta U_{i-1})$. In what follows, $\mathcal{H}_{i-1}$ and $\mathcal{H}_{\Delta U_{i-1}}$ are further explained by graphs for better understanding of their decision-making mechanisms.

First of all, let us begin with the mapping $\mathcal{H}_{i-1}$. Note that $\mathcal{H}_{i-1}$ can be completely determined by its information granules $\mathcal{H}^G_{i-1}$. So, Fig. 1 shows the graph of $\mathcal{H}_{i-1}$, where each attribute set A_{i-1j} $(j = 1, 2, \ldots, n_{i-1})$ partitions U_{i-1} into three-way decisions X_{i-1j}, Y_{i-1j} and Z_{i-1j}. Obviously, the boundary regions Z_{i-1j} $(j = 1, 2, \ldots, n_{i-1})$ need additional information to make decision.

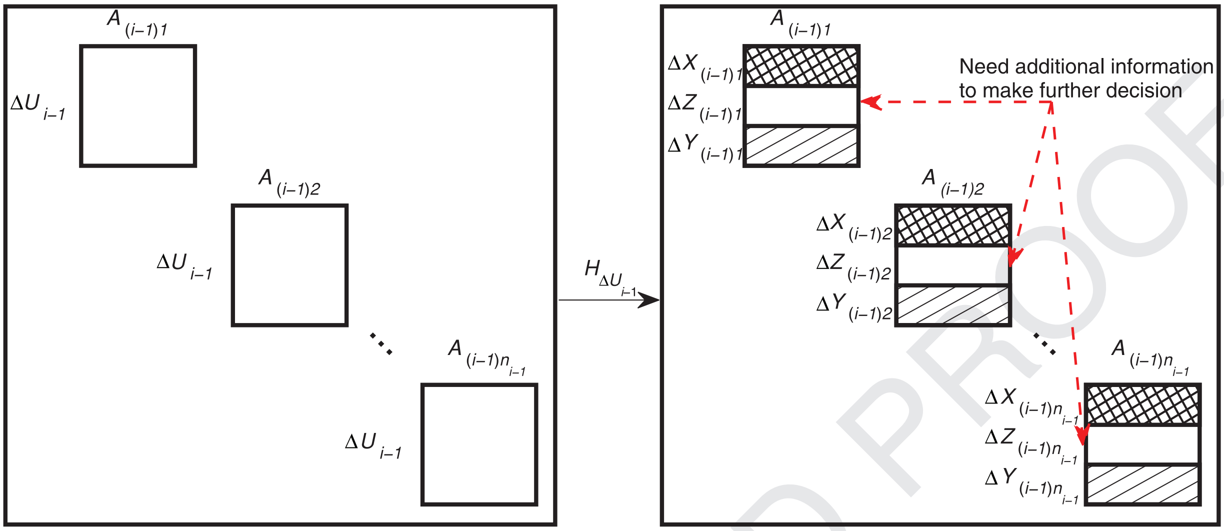

Fig. 2. The graph of $\mathcal{H}_{\Delta U_{i-1}}$.

Then, we continue to analyze the mapping $\mathcal{H}_{\Delta U_{i-1}}$. Like the case of $\mathcal{H}_{i-1}$, the information granules $\mathcal{H}^{G}_{\Delta U_{i-1}}$ can determine $\mathcal{H}_{\Delta U_{i-1}}$ with certainty. Thus, Fig. 2 in fact depicts the graph of $\mathcal{H}_{\Delta U_{i-1}}$, where each attribute set A_{i-1j} $(j = 1, 2, \ldots, n_{i-1})$ partitions ΔU_{i-1} into three-way decisions ΔX_{i-1j}, ΔY_{i-1j} and ΔZ_{i-1j}. Undoubtedly, the boundary regions ΔZ_{i-1j} $(j = 1, 2, \ldots, n_{i-1})$ also need additional information to make decision.

Furthermore, suppose that

$$\mathcal{H}_{\Delta A_{i-1}} : 2^{\mathcal{Q}(A_i)} \to \mathcal{T}(U^*) \tag{13}$$

is a mapping from $2^{\mathcal{Q}(A_i)}$ to $\mathcal{T}(U^*)$, where $\mathcal{Q}(A_{i-1}) \preceq \mathcal{Q}(A_i)$ and $U^* = \bigcup_{j=1}^{n_{i-1}} (Z_{i-1j} \cup \Delta Z_{i-1j})$. The boundary regions Z_{i-1j} and ΔZ_{i-1j} can be found in Figs. 1 and 2, respectively. In other words, the attribute set ΔA_{i-1} is combined with A_{i-1} for supporting a further decision to the objects in the boundary regions. This is in accordance with the idea of sequential or dynamic three-way decisions [9,18,61]. In addition, the graph of $\mathcal{H}_{\Delta A_{i-1}}$ is shown in Fig. 3. In the figure, each attribute set A_{ij} $(j = 1, 2, \ldots, n_{i-1})$ partitions U^* into three-way decisions $(X^{Z}_{i-1j} \cup \Delta X^{\Delta Z}_{i-1j}, Y^{Z}_{i-1j} \cup \Delta Y^{\Delta Z}_{i-1j})$, while each attribute set A_{ij} $(j = n_{i-1} + 1, \ldots, n_i)$ partitions U^* into three-way decisions (X_{ij}, Y_{ij}).

Finally, Eqs. (11)–(13) are jointly used to construct a new mapping

$$\mathcal{H}_i : 2^{\mathcal{Q}(A_i)} \to \mathcal{T}(U_i) \tag{14}$$

in which the information granules of $\mathcal{H}_i$ are defined as

$$\mathcal{H}_i(A_{it}) = \begin{cases} \mathcal{H}_{i-1}(A_{i-1s}) \cup \mathcal{H}_{\Delta U_{i-1}}(A_{i-1s}) \cup \mathcal{H}_{\Delta A_{i-1}}(A_{it}), & \text{if } \exists\, A_{i-1s} \in \mathcal{Q}(A_{i-1}) \text{ s.t. } A_{i-1s} \subseteq A_{it}, \\ \mathcal{H}_{\Delta A_{i-1}}(A_{it}), & \text{otherwise.} \end{cases} \tag{15}$$

Here, $\mathcal{H}_{\Delta U_{i-1}}(A_{i-1s})$ is set to be empty when $\Delta U_{i-1} = \emptyset$, so is $\mathcal{H}_{\Delta A_{i-1}}(A_{it})$ set when $\Delta A_{i-1} = \emptyset$.

Fig. 4 shows how to obtain the graph of $\mathcal{H}_i$ based on those of $\mathcal{H}_{i-1}$, $\mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{H}_{\Delta A_{i-1}}$. In the figure, each attribute set A_{ij} $(j = 1, 2, \ldots, n_{i-1})$ partitions U_i into three-way decisions

$$(X_{i-1j} \cup X^{Z}_{i-1j} \cup \Delta X_{i-1j} \cup \Delta X^{\Delta Z}_{i-1j}, Y_{i-1j} \cup Y^{Z}_{i-1j} \cup \Delta Y_{i-1j} \cup \Delta Y^{\Delta Z}_{i-1j}),$$

while each attribute set A_{ij} $(j = n_{i-1} + 1, \ldots, n_i)$ partitions U_i into three-way decisions (X_{ij}, Y_{ij}).

Definition 5. Let A_{i-1}, A_i be the attribute sets of $\{A_t | t \in S\}$, U_{i-1}, U_i be the object sets of $\{U_t | t \in S\}$ and $\mathcal{Q}(A_{i-1}) \preceq \mathcal{Q}(A_i)$. Denote $\Delta A_{i-1} = A_i - A_{i-1}$ and $\Delta U_{i-1} = U_i - U_{i-1}$. Suppose that $\mathcal{H}_{i-1}$, $\mathcal{L}_{i-1}$ and $\mathcal{H}_i$, $\mathcal{L}_i$ are three-way cognitive operators, where $\mathcal{H}_i$ is constructed by $\mathcal{H}_{i-1}$, $\mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{H}_{\Delta A_{i-1}}$ based on Eq. (14). Then, we say that $\mathcal{H}_i$ and $\mathcal{L}_i$ are extended three-way cognitive operators of $\mathcal{H}_{i-1}$ and $\mathcal{L}_{i-1}$ by combining the information $\mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{H}_{\Delta A_{i-1}}$.

Definition 6. Let A_{i-1}, A_i be the attribute sets of $\{A_t | t \in S\}$, U_{i-1}, U_i be the object sets of $\{U_t | t \in S\}$ and $\mathcal{Q}(A_{i-1}) \preceq \mathcal{Q}(A_i)$. Denote $\Delta A_{i-1} = A_i - A_{i-1}$ and $\Delta U_{i-1} = U_i - U_{i-1}$. Suppose that $\mathcal{H}_i$ and $\mathcal{L}_i$ are extended three-way cognitive operators of $\mathcal{H}_{i-1}$ and $\mathcal{L}_{i-1}$ by combining $\mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{H}_{\Delta A_{i-1}}$. Then, we call $\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i} = (G_{\mathcal{H}_{i-1}\mathcal{L}_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ a three-way cognitive computing state, where $G_{\mathcal{H}_{i-1}\mathcal{L}_{i-1}}$ is the set of three-way granular concepts under $\mathcal{H}_{i-1}$ and $\mathcal{L}_{i-1}$. Moreover, a collection of three-way cognitive computing states, denoted by $\mathcal{F} = \bigcup_{i=2}^{n} \{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$, is called a three-way cognitive computing system.

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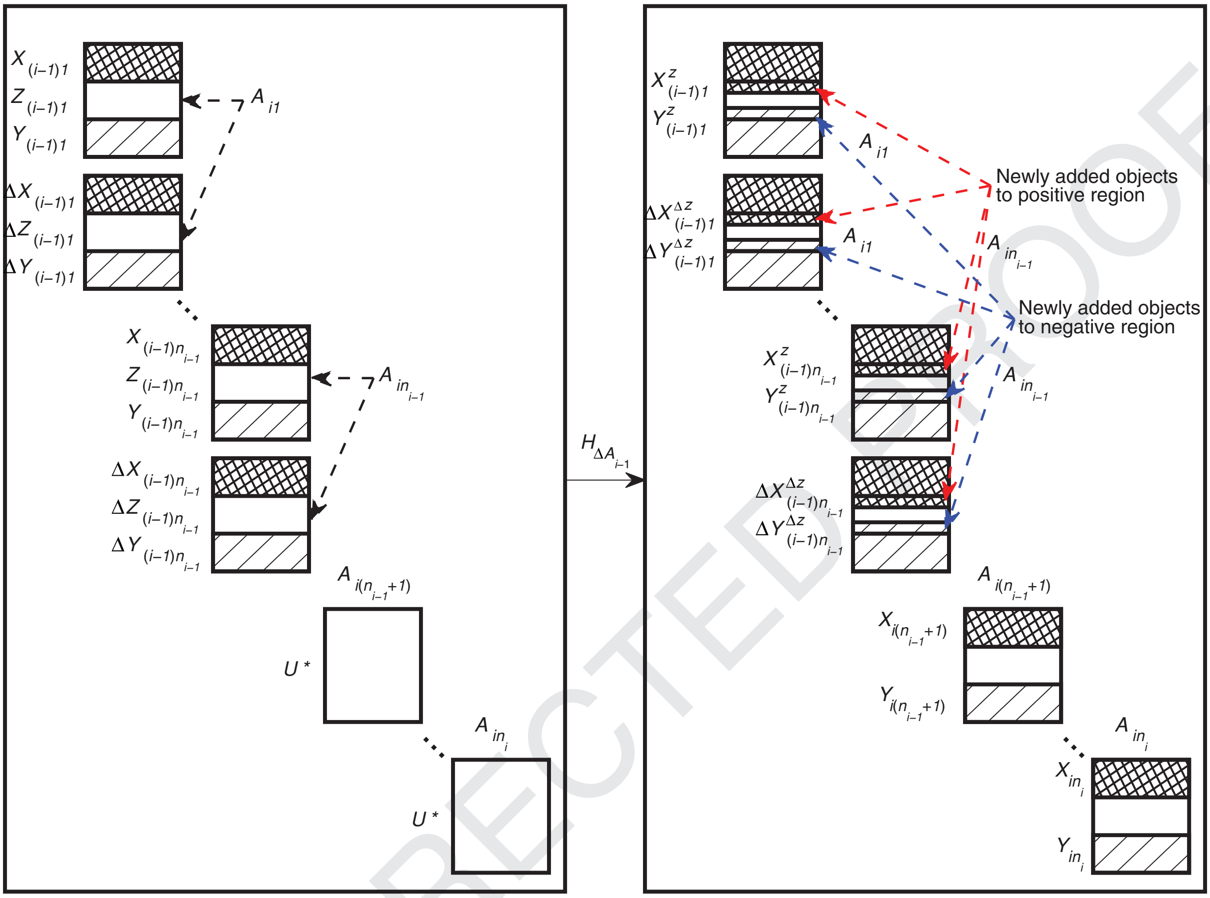

Fig. 3. The graph of $\mathcal{H}_{\Delta A_{i-1}}$.

Note that the objective of designing a three-way cognitive computing system is to update three-way granular concepts as objects and/or attributes increase in batches. This is in accordance with our common sense that in the real world, recognition of concepts will be gradually improved under the circumstance of information updating until it maintains relative stability. In order to achieve this task, it is necessary to analyze the transformation mechanism between three-way granular concepts from one three-way cognitive computing state to another.

Proposition 5. *Let $\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i} = (G_{\mathcal{H}_{i-1}\mathcal{L}_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ be a three-way cognitive computing state and $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$. For any $A_{it} \in \mathcal{Q}(A_i)$, if there exists $A_{i-1s} \in \mathcal{Q}(A_{i-1})$ such that $A_{i-1s} \subseteq A_{it}$, we have*

$$\mathcal{H}_i(A_{it}) = \mathcal{H}_{i-1}(A_{i-1s}) \cup \mathcal{H}_{\Delta U_{i-1}}(A_{i-1s}) \cup \mathcal{H}_{\Delta A_{i-1}}(A_{it}); \tag{16}$$

otherwise,

$$\mathcal{H}_i(A_{it}) = \mathcal{H}_{\Delta A_{i-1}}(A_{it}). \tag{17}$$

Proof. The proof is immediate from Eqs. (14) and (15). □

Remark 3. Based on (iii) of Definition 1, we have

$$\mathcal{L}_i\mathcal{H}_i(A_{it}) = \{A_{is} \in \mathcal{Q}(A_i) \mid \mathcal{H}_i(A_{is}) \preccurlyeq \mathcal{H}_i(A_{it})\}. \tag{18}$$

Since every $\mathcal{H}_i(A_{it})$ ($A_{it} \in \mathcal{Q}(A_i)$) can be obtained by Proposition 5, it is easy to compute each $\mathcal{L}_i\mathcal{H}_i(A_{it})$ according to Eqs. (16)–(18).

Proposition 5 and Remark 3 can be jointly used to achieve the task of transforming the information granules $G_{\mathcal{H}_{i-1}\mathcal{L}_{i-1}}$ to $G_{\mathcal{H}_i\mathcal{L}_i}$.

Note that for a given three-way cognitive computing system $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$, all information granules $G_{\mathcal{H}_1\mathcal{L}_1}, G_{\mathcal{H}_2\mathcal{L}_2}, \ldots, G_{\mathcal{H}_n\mathcal{L}_n}$ are unknown in advance. Their sequential computation processes are described below:

(i) three-way cognitive operators $\mathcal{H}_1$ and $\mathcal{L}_1$ are used to compute $G_{\mathcal{H}_1\mathcal{L}_1}$ according to Eq. (10),

(ii) and then recursive strategy is adopted to generate $G_{\mathcal{H}_2\mathcal{L}_2}, \ldots, G_{\mathcal{H}_n\mathcal{L}_n}$ in sequence based on Proposition 5 and Remark 3.

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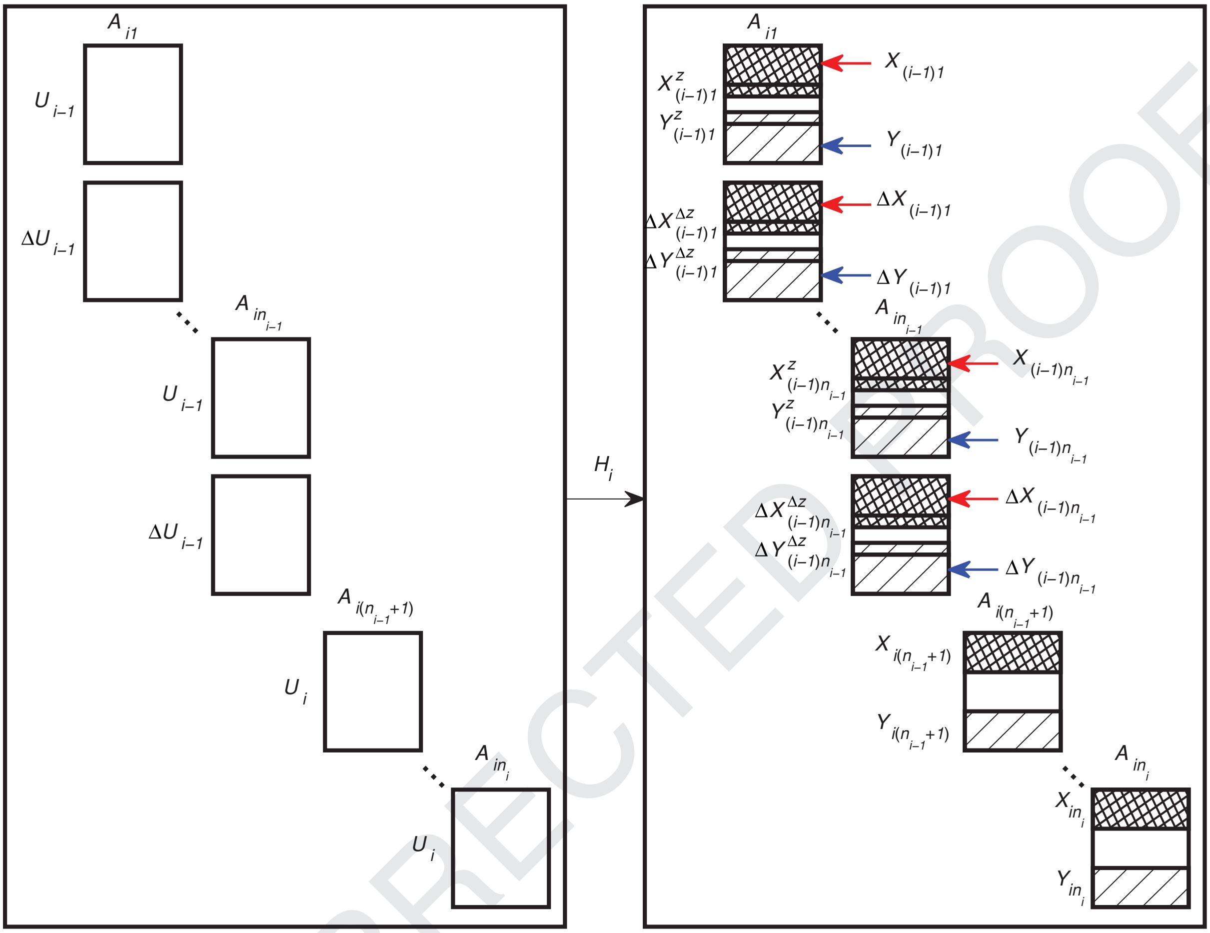

Fig. 4. The graph of $\mathcal{H}_i$.

Moreover, Algorithm 1 describes the detailed procedure of solving this problem. The time complexity is analyzed as follows. Suppose that $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$ is the input three-way cognitive computing system. Running Step 1 takes $O(|A_1|^2|U_1|)$ based on Eqs. (10) and (18). The time complexity of Steps 3–11 is $O(|A_i|(|A_i|^2+|U_i|))$, and that of Steps 12–24 is $O(|A_i|^2|U_i|)$. As a result, the time complexity of Algorithm 1 is $O(n|A_n|^2(|A_n|+|U_n|))$, where n is the number of three-way cognitive computing states. Obviously, its time complexity is polynomial.

Finally, we use an example to illustrate Algorithm 1. In order to make the example better understood, we give below a sufficient and necessary condition to the preparatory work of generating $G_{\mathcal{H}_i\mathcal{L}_i}$.

A three-way cognitive computing state $\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i} = (G_{\mathcal{H}_{i-1}\mathcal{L}_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ can be constructed to obtain three-way granular concepts $G_{\mathcal{H}_i\mathcal{L}_i}$ via Eqs. (16)–(18)if and only if the following conditions are prepared:

(a) objects and attributes are updated;
(b) the evaluation function f_{A_i} and the thresholds α and β are properly defined;
(c) $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$ is satisfied;
(d) three-way granular concepts $G_{\mathcal{H}_{i-1}\mathcal{L}_{i-1}}$ of the previous state are known;
(e) information granules of $\mathcal{H}_{i-1}$, $\mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{H}_{\Delta A_{i-1}}$ are computed.

Example 4. In Example 1, nine manuscripts were evaluated by seven reviewers. As time goes by, on one hand, new manuscripts will arrive. On the other hand, those falling into the boundary regions need to be further evaluated by inviting additional reviewers. It is supposed that the information updating on the manuscripts and reviewers is shown in Table 2. That is, new manuscripts x_{10}, x_{11}, x_{12}, x_{13} and x_{14} were submitted to the reviewing dataset, and at the same time, new reviewers r_8, r_9, r_{10} and r_{11} were invited to evaluate the manuscripts falling into the boundary regions. It can be observed from Table 2 that the manuscript x_{10} falls into Domain 2, the manuscript x_{11} Domain 1, the manuscript x_{12} Domain 3, and others a new domain (i.e. Domain 4). Among these newly invited reviewers, r_8 is from Domain 1, r_9 is from Domain 2, and r_{10} and r_{11} are from Domain 4. Since new manuscripts and reviewers have been added, it is necessary to update the three-way granular concepts which can be found in Example 3.

Algorithm 1 Computing three-way granular concepts of a three-way cognitive computing system.

Require: $\mathcal{F} = \bigcup_{i=2}^{n} \{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$, where $\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i} = (G_{\mathcal{H}_{i-1}\mathcal{L}_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ is a three-way cognitive computing state with $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$.
Ensure: Three-way granular concepts $G_{\mathcal{H}_n\mathcal{L}_n}$ of $\mathcal{F}$.
Initialize $G_{\mathcal{H}_1\mathcal{L}_1} = \{\langle \mathcal{H}_1(A_{1s}), \mathcal{L}_1\mathcal{H}_1(A_{1s})\rangle \mid A_{1s} \in \mathcal{Q}(A_1)\}$ and $i = 2$;
While $i \leq n$
  Set $\Omega_1 = \emptyset$;
  For each $A_{it} \in \mathcal{Q}(A_i)$
    If there exists $A_{i-1s} \in \mathcal{Q}(A_{i-1})$ such that $A_{i-1s} \subseteq A_{it}$
      let $\mathcal{H}_i(A_{it}) = \mathcal{H}_{i-1}(A_{i-1s}) \cup \mathcal{H}_{\Delta U_{i-1}}(A_{i-1s}) \cup \mathcal{H}_{\Delta A_{i-1}}(A_{it})$;
    Else
      let $\mathcal{H}_i(A_{it}) = \mathcal{H}_{\Delta A_{i-1}}(A_{it})$;
    End If
    Set $\Omega_1 \leftarrow \Omega_1 \cup \{\mathcal{H}_i(A_{it})\}$;
  End For
  Set $\Omega_2 = \emptyset$;
  For each $A_{it} \in \mathcal{Q}(A_i)$
    Let $B = \emptyset$
    For each $\mathcal{H}_i(A_{is}) \in \Omega_1$
      If $\mathcal{H}_i(A_{is}) \preccurlyeq \mathcal{H}_i(A_{it})$
        do $B \leftarrow B \cup \{A_{is}\}$;
      End If
    End For
    Set $\mathcal{L}_i\mathcal{H}_i(A_{it}) = B$;
    Do $\Omega_2 \leftarrow \Omega_2 \cup \{\mathcal{L}_i\mathcal{H}_i(A_{it})\}$;
  End For
  Compute $G_{\mathcal{H}_i\mathcal{L}_i}$ based on Ω_1 and Ω_2;
  $i \leftarrow i + 1$;
End While
Return $G_{\mathcal{L}_n\mathcal{H}_n}$.

Table 2
A reviewing dataset with information updating on manuscripts and reviewers.

U_2	Domain 1			Domain 2			Domain 3			Domain 4	
	r_1	r_2	r_8	r_3	r_4	r_9	r_5	r_6	r_7	r_{10}	r_{11}
x_1	Accept	Accept									
x_2	Accept	Reject	Reject								
x_3	Reject	Accept	Reject								
x_4				Accept	Reject	Accept					
x_5				Accept	Accept						
x_6				Reject	Reject						
x_7							Accept	Reject	Accept		
x_8							Accept	Accept	Accept		
x_9							Reject	Reject	Accept		
x_{10}				Accept	Accept						
x_{11}	Reject	Reject									
x_{12}							Reject	Accept	Reject		
x_{13}										Reject	Accept
x_{14}										Accept	Accept

Similar to the case in Example 1, we also take $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$. The evaluation function $f_{A_{it}}(x)$ ($A_{it} \in \mathcal{Q}(A_i)$) is the ratio of the number of Accepts given to x to that of reviewers assigned to x under the columns A_{it}.

From Example 1, it follows $U_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$, $A_1 = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$, $A_{11} = \{r_1, r_2\}$, $A_{12} = \{r_3, r_4\}$, $A_{13} = \{r_5, r_6, r_7\}$ and $\mathcal{Q}(A_1) = \{A_{11}, A_{12}, A_{13}\}$. Let $U_2 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\}$, $A_2 = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}\}$, $A_{21} = \{r_1, r_2, r_8\}$, $A_{22} = \{r_3, r_4, r_9\}$, $A_{23} = \{r_5, r_6, r_7\}$, $A_{24} = \{r_{10}, r_{11}\}$ and $\mathcal{Q}(A_2) = \{A_{21}, A_{22}, A_{23}, A_{24}\}$. Then, $\mathcal{Q}(A_1) \leq \mathcal{Q}(A_2)$ is satisfied. Moreover, we denote $\Delta U_1 = U_2 - U_1 = \{x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\}$ and $\Delta A_1 = A_2 - A_1 = \{r_8, r_9, r_{10}, r_{11}\}$.

It can be seen from Example 3 that three-way granular concepts under $\mathcal{H}_1$ and $\mathcal{L}_1$ are as follows:

$$G_{\mathcal{H}_1\mathcal{L}_1} = \{\langle(\{x_1\}, \emptyset), \{A_{11}\}\rangle, \langle(\{x_5\}, \{x_6\}), \{A_{12}\}\rangle, \langle(\{x_7, x_8\}, \{x_9\}), \{A_{13}\}\rangle\}$$

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from which we can obtain the information granules of $\mathcal{H}_1$.

Furthermore, we know from Table 2 that the information granules of $\mathcal{H}_{\Delta U_1}$ are

$$\mathcal{H}^G_{\Delta U_1} = \{\{A_{11}\} \mapsto (\emptyset, \{x_{11}\}), \{A_{12}\} \mapsto (\{x_{10}\}, \emptyset), \{A_{13}\} \mapsto (\emptyset, \{x_{12}\})\},$$

and those of $\mathcal{H}_{\Delta A_1}$ are

$$\mathcal{H}^G_{\Delta A_1} = \{\{A_{21}\} \mapsto (\emptyset, \{x_2, x_3\}), \{A_{22}\} \mapsto (\{x_4\}, \emptyset), \{A_{23}\} \mapsto (\emptyset, \emptyset), \{A_{24}\} \mapsto (\{x_{14}\}, \emptyset)\}.$$

To sum up, all the conditions (a)–(e) have been satisfied. According to the statement in the paragraph above Example 4, a three-way cognitive computing state $\mathcal{F}_{\mathcal{H}_2\mathcal{L}_2} = (G_{\mathcal{H}_1\mathcal{L}_1}, \mathcal{H}_{\Delta U_1}, \mathcal{H}_{\Delta A_1})$ can be constructed to obtain three-way granular concepts $G_{\mathcal{H}_2\mathcal{L}_2}$ based on Eqs. (16)–(18). The detailed computations are given below:

$$\begin{aligned}
&\mathcal{H}_2(A_{21}) = \mathcal{H}_1(A_{11}) \cup \mathcal{H}_{\Delta U_1}(A_{11}) \cup \mathcal{H}_{\Delta A_1}(A_{21}) = (\{x_1\}, \{x_2, x_3, x_{11}\}) \text{ due to } A_{11} \subseteq A_{21},\\
&\mathcal{H}_2(A_{22}) = \mathcal{H}_1(A_{12}) \cup \mathcal{H}_{\Delta U_1}(A_{12}) \cup \mathcal{H}_{\Delta A_1}(A_{22}) = (\{x_4, x_5, x_{10}\}, \{x_6\}) \text{ due to } A_{12} \subseteq A_{22},\\
&\mathcal{H}_2(A_{23}) = \mathcal{H}_1(A_{13}) \cup \mathcal{H}_{\Delta U_1}(A_{13}) \cup \mathcal{H}_{\Delta A_1}(A_{23}) = (\{x_7, x_8\}, \{x_9, x_{12}\}) \text{ due to } A_{13} \subseteq A_{23},\\
&\mathcal{H}_2(A_{24}) = \mathcal{H}_{\Delta A_1}(A_{24}) = (\{x_{14}\}, \emptyset) \text{ since there does not exist } A_{1s} \subseteq A_{24} \text{ such that } A_{1s} \subseteq A_{24},
\end{aligned}$$

and

$$\mathcal{L}_2\mathcal{H}_2(A_{21}) = \{A_{21}\},\ \mathcal{L}_2\mathcal{H}_2(A_{22}) = \{A_{22}\},\ \mathcal{L}_2\mathcal{H}_2(A_{23}) = \{A_{23}\},\ \mathcal{L}_2\mathcal{H}_2(A_{24}) = \{A_{24}\}.$$

Consequently, we have

$$G_{\mathcal{H}_2\mathcal{L}_2} = \{\langle(\{x_1\}, \{x_2, x_3, x_{11}\}), \{A_{21}\}\rangle, \langle(\{x_4, x_5, x_{10}\}, \{x_6\}), \{A_{22}\}\rangle, \langle(\{x_7, x_8\}, \{x_9, x_{12}\}), \{A_{23}\}\rangle, \langle(\{x_{14}\}, \emptyset), \{A_{24}\}\rangle\}.$$

Three-way decisions derived by the granular concepts $G_{\mathcal{H}_2\mathcal{L}_2}$ are as follows:

- $\langle(\{x_1\}, \{x_2, x_3, x_{11}\}), \{A_{21}\}\rangle$: according to the reviewers r_1, r_2 and r_8 from Domain 1, manuscript x_1 is accepted, while x_2, x_3 and x_{11} are rejected;
- $\langle(\{x_4, x_5, x_{10}\}, \{x_6\}), \{A_{22}\}\rangle$: according to the reviewers r_3, r_4 and r_9 from Domain 2, manuscripts x_4, x_5 and x_{10} are accepted, while x_6 is rejected;
- $\langle(\{x_7, x_8\}, \{x_9, x_{12}\}), \{A_{23}\}\rangle$: according to the reviewers r_5, r_6 and r_7 from Domain 3, manuscripts x_7 and x_8 are accepted, while x_9 and x_{12} are rejected;
- $\langle(\{x_{14}\}, \emptyset), \{A_{24}\}\rangle$: according to the reviewers r_{10} and r_{11} from Domain 4, manuscript x_{14} is accepted.

Furthermore, we point out that these three-way granular concepts can also be used in cognitive concept learning (see Examples 5 and 6 in the next section for details).

4. Cognitive processes

From cognitive computing [42], the obtained three-way granular concepts of a three-way cognitive computing system can be further used to learn three-way cognitive concepts from a given clue. Note that the clue may be three-way decisions, a set of attribute classes or both of them. Generally speaking, deriving new cognitive concepts from a given clue by induction, approximation or reasoning is called the cognitive process. For instance, we take Example 4 to describe the scenarios. Suppose that $(\{x_1, x_4, x_5, x_7, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}\})$ is an available clue. Then what knowledge are such three-way decisions induced by? It is easy to observe from Example 4 that there is no a direct answer to this question since no extent of a three-way granular concept is exactly $(\{x_1, x_4, x_5, x_7, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}\})$. In what follows, we try to find the answers for this kind of questions.

Considering that the idea of lower and upper approximations in rough set theory has been widely applied to concept approximation [14,37,38], we use this idea to simulate cognitive processes.

In rough set theory [25], an *information system* is represented as $I = (U, A)$ in which each object $x \in U$ has a value $a(x)$ under every attribute $a \in A$.

For a nonempty subset $A_i \subseteq A$, an equivalence relation $IND(A_i)$ is defined by

$$IND(A_i) = \{(x, y) \in U \times U \mid a(x) = a(y) \text{ for all } a \in A_i\}.$$

In fact, $IND(A_i)$ can induce a partition $U/IND(A_i)$ of U by taking each equivalence class as $[x]_{A_i} = \{y \in U \mid (x, y) \in IND(A_i)\}$. That is, $U/IND(A_i) = \{[x]_{A_i} \mid x \in U\}$. Then, for any target set $X \subseteq U$, its lower and upper approximations are respectively defined as

$$\underline{A_i}(X) = \bigcup_{Y \in U/IND(A_i), Y \subseteq X} Y \quad \text{and} \quad \overline{A_i}(X) = \bigcup_{Y \in U/IND(A_i), Y \cap X \neq \emptyset} Y,$$

and the pair $[\underline{A_i}(X), \overline{A_i}(X)]$ is called a rough set of X with respect to A_i. The relationship between X and its lower and upper approximations is shown in Fig. 5.

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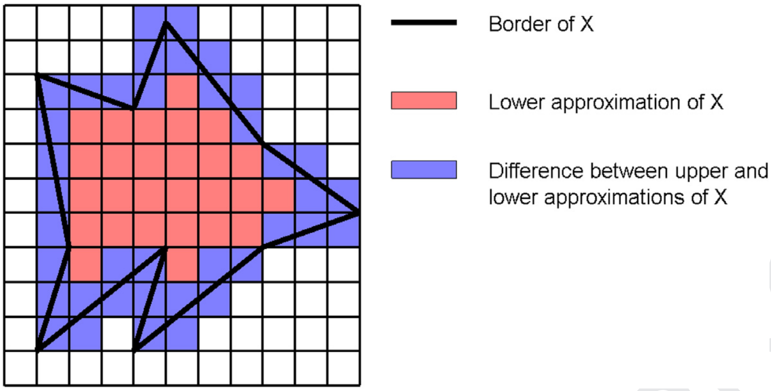

Fig. 5. The relationship among X, $\underline{A_i}(X)$ and $\overline{A_i}(X)$.

4.1. Three-way cognitive concept learning from three-way decisions

Based on the idea of lower and upper approximations, we put forward below an approach to learn an exact or two approximate three-way cognitive concepts from three-way decisions.

Definition 7. Let $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$ be a three-way cognitive computing system and $G_{\mathcal{H}_n\mathcal{L}_n}$ be three-way granular concepts of $\mathcal{F}$. Then, the lower and upper approximations of three-way decisions (X_0, Y_0) are respectively defined as

$$\underline{\text{Apr}}(X_0, Y_0) = \bigcup_{\langle (X,Y),B\rangle \in G_{\mathcal{H}_n\mathcal{L}_n}, (X,Y)\preccurlyeq (X_0,Y_0)} (X, Y) \quad \text{and} \quad \overline{\text{Apr}}(X_0, Y_0) = \bigcup_{\langle (X,Y),B\rangle \in G_{\mathcal{H}_n\mathcal{L}_n}, (X,Y)\cap (X_0,Y_0)\neq(\emptyset,\emptyset)} (X, Y). \tag{19}$$

Proposition 6. *Both* $\underline{\text{Apr}}(X_0, Y_0)$ *and* $\overline{\text{Apr}}(X_0, Y_0)$ *are extents of three-way cognitive concepts.*

Proof. Let $\langle (X_i, Y_i), B_i\rangle, \langle (X_j, Y_j), B_j\rangle \in G_{\mathcal{H}_n\mathcal{L}_n}$. Then, by Eq. (8), $\langle (X_i, Y_i) \cup (X_j, Y_j), \mathcal{L}_n\mathcal{H}_n(B_i \cup B_j)\rangle$ is a three-way cognitive concept, which means that $(X_i, Y_i)\cup(X_j, Y_j)$ must be an extent of a three-way cognitive concept. Furthermore, by mathematical induction, we can prove that both $\underline{\text{Apr}}(X_0, Y_0)$ and $\overline{\text{Apr}}(X_0, Y_0)$ are extents of three-way cognitive concepts. □

Definition 8. For three-way decisions (X_0, Y_0), we call

$$\langle \underline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\underline{\text{Apr}}(X_0, Y_0))\rangle \text{ and } \langle \overline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\overline{\text{Apr}}(X_0, Y_0))\rangle$$

the learnt three-way cognitive concepts from (X_0, Y_0). Moreover, the learning accuracy is defined as

$$\alpha(X_0, Y_0) = 1 - \frac{|\overline{\text{Apr}}(X_0, Y_0) - \underline{\text{Apr}}(X_0, Y_0)|}{2|U_n|},$$

where $|(\bullet, \bullet)|$ is the total number of elements in the first and the second sets.

From Definition 8, we know that $\alpha(X_0, Y_0) = 1$ if and only if $\overline{\text{Apr}}(X_0, Y_0) = \underline{\text{Apr}}(X_0, Y_0)$. In this case, we can learn an exact three-way cognitive concept; otherwise, two approximate three-way cognitive concepts are learnt. Algorithm 2 gives the detailed procedure to learn cognitive concept(s) from three-way decisions.

According to Eq. (19), Steps 2–10 in Algorithm 2 are to compute the lower and upper approximations of three-way decisions (X_0, Y_0). Furthermore, Step 11 is to find an exact or two approximate three-way cognitive concepts for (X_0, Y_0) as well as the learning accuracy $\alpha(X_0, Y_0)$. So, the time complexity of Algorithm 2 is $O(|U_n||A_n|)$.

Example 5 (Continued with Example 4)**.** Suppose that the manuscripts x_1, x_4, x_5, x_7 and x_{10} were accepted, while x_2, x_3, x_6, x_9 and x_{11} were rejected. Then which domain are the reviewers (making such three-way decisions) from? To answer this question, it needs to learn three-way cognitive concepts from $X_0 = \{x_1, x_4, x_5, x_7, x_{10}\}$ and $Y_0 = \{x_2, x_3, x_6, x_9, x_{11}\}$ based on the granular concepts $G_{\mathcal{L}_2\mathcal{H}_2}$. By Eq. (19), it follows that:

$$\begin{aligned}\underline{\text{Apr}}(X_0, Y_0) &= \bigcup_{\langle (X,Y),B\rangle \in G_{\mathcal{H}_2\mathcal{L}_2}, (X,Y)\preccurlyeq (X_0,Y_0)} (X, Y)\\ &= (\{x_1\}, \{x_2, x_3, x_{11}\}) \cup (\{x_4, x_5, x_{10}\}, \{x_6\})\\ &= (\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\}),\\ \overline{\text{Apr}}(X_0, Y_0) &= \bigcup_{\langle (X,Y),B\rangle \in G_{\mathcal{H}_2\mathcal{L}_2}, (X,Y)\cap (X_0,Y_0)\neq(\emptyset,\emptyset)} (X, Y)\\ &= (\{x_1\}, \{x_2, x_3, x_{11}\}) \cup (\{x_4, x_5, x_{10}\}, \{x_6\}) \cup (\{x_7, x_8\}, \{x_9, x_{12}\})\end{aligned}$$

Algorithm 2 Three-way cognitive concept learning from three-way decisions.

Require: Three-way granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a three-way cognitive computing system $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$ and three-way decisions (X_0, Y_0).
Ensure: An exact or two approximate three-way cognitive concepts with the learning accuracy for (X_0, Y_0).

```
Initialize Π = ∅, Ω = ∅, and label the elements of G_{L_n H_n} as
    ⟨H_n(A_n1), L_n H_n(A_n1)⟩, ⟨H_n(A_n2), L_n H_n(A_n2)⟩, ..., ⟨H_n(A_nt), L_n H_n(A_nt)⟩;
For each i ∈ {1, 2, ..., t}
    If H_n(A_ni) ≼ (X_0, Y_0)
        Π ← Π ∪ {⟨H_n(A_ni), L_n H_n(A_ni)⟩};
    End If
    If H_n(A_ni) ∩ (X_0, Y_0) ≠ (∅, ∅)
        Ω ← Ω ∪ {⟨H_n(A_ni), L_n H_n(A_ni)⟩};
    End If
End For
Set Apr_(X_0, Y_0) = ⋃_{⟨(X,Y),B⟩∈Π} (X, Y) and Apr¯(X_0, Y_0) = ⋃_{⟨(X,Y),B⟩∈Ω} (X, Y);
Compute B_0_ = L_n(Apr_(X_0, Y_0)), B¯_0 = L_n(Apr¯(X_0, Y_0)) and α(X_0, Y_0) = 1 − |Apr¯(X_0,Y_0) − Apr_(X_0,Y_0)| / (2|U_n|);
Return ⟨Apr_(X_0, Y_0), B_0_⟩, ⟨Apr¯(X_0, Y_0), B¯_0⟩ and α(X_0, Y_0).
```

$$= (\{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\}).$$

So, three-way cognitive concepts

$$\langle(\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\}), \{A_{21}, A_{22}\}\rangle \text{ and } \langle(\{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\}), \{A_{21}, A_{22}, A_{23}\}\rangle$$

are learnt from (X_0, Y_0) with the learning accuracy $\alpha(X_0, Y_0) = \frac{6}{7}$. As a result, there does not exist any domain that the reviewers making three-way decisions (X_0, Y_0) are from. However, the reviewers from Domains 1 and 2 made three-way decisions which are decision-consistent with respect to (X_0, Y_0), and (X_0, Y_0) is decision-consistent with respect to three-way decisions made by the reviewers from Domains 1–3.

4.2. Three-way cognitive concept learning from a set of attribute classes

Similar to the discussion in Section 4.1, we continue to learn an exact or two approximate three-way cognitive concepts from a set of attribute classes.

Definition 9. Let $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$ be a three-way cognitive computing system and $G_{\mathcal{H}_n\mathcal{L}_n}$ be three-way granular concepts of $\mathcal{F}$. Then, the lower and upper approximations of $B_0 \in 2^{\mathcal{Q}(A_n)}$ are respectively defined as

$$\underline{\mathrm{Apr}}(B_0) = \mathcal{L}_n\mathcal{H}_n\left(\bigcup_{\langle(X,Y),B\rangle\in G_{\mathcal{H}_n\mathcal{L}_n}, B\subseteq B_0} B\right) \text{ and } \overline{\mathrm{Apr}}(B_0) = \mathcal{L}_n\mathcal{H}_n\left(\bigcup_{\langle(X,Y),B\rangle\in G_{\mathcal{H}_n\mathcal{L}_n}, B\cap B_0\neq\emptyset} B\right). \tag{20}$$

Proposition 7. *Both* $\underline{\mathrm{Apr}}(B_0)$ *and* $\overline{\mathrm{Apr}}(B_0)$ *are intents of three-way cognitive concepts.*

Proof. The proof is obvious from Proposition 5 and Eq. (20). □

Definition 10. For any $B_0 \in 2^{\mathcal{Q}(A_n)}$, we call

$$\langle\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)\rangle \text{ and } \langle\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)\rangle$$

the learnt three-way cognitive concepts from B_0. Moreover, the learning accuracy is defined as

$$\beta(B_0) = 1 - \frac{|\overline{\mathrm{Apr}}(B_0) - \underline{\mathrm{Apr}}(B_0)|}{|A_n|}.$$

From Definition 10, we know that $\beta(B_0) = 1$ if and only if $\overline{\mathrm{Apr}}(B_0) = \underline{\mathrm{Apr}}(B_0)$. In this case, we can learn an exact three-way cognitive concept; otherwise, two approximate three-way cognitive concepts are learnt. Algorithm 3 gives the detailed procedure to learn three-way cognitive concept(s) from a set of attribute classes.

According to Eq. (20), Steps 2–10 in Algorithm 3 are to compute the lower and upper approximations of B_0. Furthermore, Step 11 is to find an exact or two approximate three-way cognitive concepts for B_0 as well as the learning accuracy $\beta(B_0)$. So, the time complexity of Algorithm 3 is $O(|U_n||A_n|)$.

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Algorithm 3 Three-way cognitive concept learning from a set of attribute classes.

Require: Three-way granular concepts$G_{\mathcal{L}_n\mathcal{H}_n}$ of a three-way cognitive computing system $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$ and a set of attribute classes B_0.

Ensure: An exact or two approximate three-way cognitive concepts with the learning accuracy for B_0.

Initialize $\Pi = \emptyset$, $\Omega = \emptyset$, and label the elements of $G_{\mathcal{L}_n\mathcal{H}_n}$ as $\langle\mathcal{H}_n(A_{n1}), \mathcal{L}_n\mathcal{H}_n(A_{n1})\rangle, \langle\mathcal{H}_n(A_{n2}), \mathcal{L}_n\mathcal{H}_n(A_{n2})\rangle, \ldots, \langle\mathcal{H}_n(A_{nt}), \mathcal{L}_n\mathcal{H}_n(A_{nt})\rangle$;
For each $i \in \{1, 2, \ldots, t\}$
  If $\mathcal{L}_n\mathcal{H}_n(A_{ni}) \subseteq B_0$
    $\Pi \leftarrow \Pi \cup \{\langle\mathcal{H}_n(A_{ni}), \mathcal{L}_n\mathcal{H}_n(A_{ni})\rangle\}$;
  End If
  If $\mathcal{L}_n\mathcal{H}_n(A_{ni}) \cap B_0 \neq \emptyset$
    $\Omega \leftarrow \Omega \cup \{\langle\mathcal{H}_n(A_{ni}), \mathcal{L}_n\mathcal{H}_n(A_{ni})\rangle\}$;
  End If
End For
Set$\underline{\mathrm{Apr}}(B_0) = \mathcal{L}_n\mathcal{H}_n\left(\bigcup_{\langle(X,Y),B\rangle\in\Pi} B\right)$ and $\overline{\mathrm{Apr}}(B_0) = \mathcal{L}_n\mathcal{H}_n\left(\bigcup_{\langle(X,Y),B\rangle\in\Omega} B\right)$;
Compute$(\underline{X}_0, \underline{Y}_0) = \mathcal{H}_n(\underline{\mathrm{Apr}}(B_0))$,$(\overline{X}_0, \overline{Y}_0) = \mathcal{H}_n(\overline{\mathrm{Apr}}(B_0))$ and $\beta(B_0) = 1 - \frac{|\overline{\mathrm{Apr}}(B_0) - \underline{\mathrm{Apr}}(B_0)|}{|A_n|}$;
Return$\langle(\underline{X}_0, \underline{Y}_0), \underline{\mathrm{Apr}}(B_0)\rangle$,$\langle(\overline{X}_0, \overline{Y}_0), \overline{\mathrm{Apr}}(B_0)\rangle$and $\beta(B_0)$.

Example 6 (Continued with Example 4)**.** Which manuscripts were accepted and which ones were rejected by the reviewers r_1, r_2, r_3, r_4, r_8 and r_9 from Domains 1 and 2? Since $A_{21} = \{r_1, r_2, r_8\}$ and $A_{22} = \{r_3, r_4, r_9\}$, then $B_0 = \{A_{21}, A_{22}\}$ represents the reviewers under consideration. Moreover, to answer the above question, it needs to learn three-way cognitive concept(s) from B_0 because there is no granular concept in $G_{\mathcal{L}_2\mathcal{H}_2}$ with its intent being B_0 exactly. By Eq. (20), we have

$$\begin{aligned}\underline{\mathrm{Apr}}(B_0) &= \mathcal{L}_2\mathcal{H}_2\left(\bigcup_{\langle(X,Y),B\rangle\in G_{\mathcal{H}_2\mathcal{L}_2}, B\subseteq B_0} B\right)\\ &= \mathcal{L}_2\mathcal{H}_2(\{A_{21}\}\cup\{A_{22}\})\\ &= \{A_{21}, A_{22}\},\\ \overline{\mathrm{Apr}}(B_0) &= \mathcal{L}_2\mathcal{H}_2\left(\bigcup_{\langle(X,Y),B\rangle\in G_{\mathcal{H}_2\mathcal{L}_2}, B\cap B_0\neq\emptyset} B\right)\\ &= \mathcal{L}_2\mathcal{H}_2(\{A_{21}\}\cup\{A_{22}\})\\ &= \{A_{21}, A_{22}\}.\end{aligned}$$

Thus, we find an exact three-way cognitive concept $\langle(\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\}), \{A_{21}, A_{22}\}\rangle$ from B_0 with the learning accuracy $\beta(B_0) = 1$. In other words, the manuscripts x_1, x_4, x_5, x_{10} were accepted, while x_2, x_3, x_6 and x_{11} were rejected by the reviewers from Domains 1 and 2.

4.3. Three-way cognitive concept learning from three-way decisions and a set of attribute classes

In the previous Sections 4.1 and 4.2, we have discussed the case of learning three-way cognitive concepts from three-way decisions as well as a set of attribute classes. In the real world, however, it may be encountered that three-way decisions and a set of attribute classes are available simultaneously. This issue is investigated below.

Definition 11. Let $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$ be a three-way cognitive computing system and $G_{\mathcal{H}_n\mathcal{L}_n}$ be three-way granular concepts of $\mathcal{F}$. For three-way decisions (X_0, Y_0) and $B_0 \in 2^{\mathcal{Q}(A_n)}$, if $\underline{\mathrm{Apr}}(X_0, Y_0) \preccurlyeq \mathcal{H}_n(B_0) \preccurlyeq \overline{\mathrm{Apr}}(X_0, Y_0)$ and $\overline{\mathrm{Apr}}(B_0) \subseteq \mathcal{L}_n(X_0, Y_0) \subseteq \underline{\mathrm{Apr}}(B_0)$, we say that (X_0, Y_0) and B_0 are jointly concept-inducible; otherwise, we say that they are jointly concept-uninducible.

Definition 11 divides the pairs of (X_i, Y_i) and B_i ($i \in S$) into two categories: concept-inducible and concept-uninducible pairs. In what follows, we only discuss concept-inducible pairs since concept-uninducible ones are less related to each other.

Example 7 (Continued with Examples 5 and 6)**.** In Example 5, $X_0 = \{x_1, x_4, x_5, x_7, x_{10}\}$, $Y_0 = \{x_2, x_3, x_6, x_9, x_{11}\}$, $\underline{\mathrm{Apr}}(X_0, Y_0) = (\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\})$ and $\overline{\mathrm{Apr}}(X_0, Y_0) = (\{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\})$. By (iii) of Definition 1, we have $\mathcal{L}_2(X_0, Y_0) = \{A_{21}, A_{22}\}$. Moreover, in Example 6, $B_0 = \{A_{21}, A_{22}\}$, $\underline{\mathrm{Apr}}(B_0) = \{A_{21}, A_{22}\}$, $\overline{\mathrm{Apr}}(B_0) = \{A_{21}, A_{22}\}$ and $\mathcal{H}_2(B_0) = (\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\})$. Then, according to Definition 11, we know that (X_0, Y_0) and B_0 are jointly concept-inducible.

Now, we discuss how to learn three-way cognitive concepts from concept-inducible pairs.

Definition 12. Let $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$ be a three-way cognitive computing system and $G_{\mathcal{H}_n\mathcal{L}_n}$ be three-way granular concepts of $\mathcal{F}$. For three-way decisions (X_0, Y_0) and $B_0 \in 2^{\mathcal{Q}(A_n)}$, if they are jointly concept-inducible, we call

$$\langle \underline{\mathrm{Apr}}(X_0, Y_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0, Y_0))\rangle \bigwedge \langle \mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)\rangle \tag{21}$$

and

$$\langle \overline{\mathrm{Apr}}(X_0, Y_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0, Y_0))\rangle \bigvee \langle \mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)\rangle \tag{22}$$

the learnt three-way cognitive concepts from the pair of (X_0, Y_0) and B_0. Furthermore, the learning accuracy is defined as

$$\gamma((X_0, Y_0), B_0) = min\{\alpha(X_0, Y_0), \beta(B_0)\}.$$

From Definition 12, we know that $\gamma((X_0, Y_0), B_0) = 1$ if and only if $\overline{\mathrm{Apr}}(X_0, Y_0) = \underline{\mathrm{Apr}}(X_0, Y_0)$ and $\overline{\mathrm{Apr}}(B_0) = \underline{\mathrm{Apr}}(B_0)$. In this case, we learn an exact three-way cognitive concept; otherwise, two approximate three-way cognitive concepts are learnt. Algorithm 4 shows the detailed procedure to learn three-way cognitive concept(s) from three-way decisions and a set of attribute classes.

Algorithm 4 Cognitive concept learning from three-way decisions and a set of attribute classes.

Require: Three-way granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a three-way cognitive computing system $\mathcal{F} = \bigcup_{i=2}^{n}\{\mathcal{F}_{\mathcal{H}_i\mathcal{L}_i}\}$ and the pair of (X_0, Y_0) and B_0.
Ensure: Three-way cognitive concept(s) learnt from the concept-inducible pair of (X_0, Y_0) and B_0.
Call Algorithm 2 to learn $\langle \underline{\mathrm{Apr}}(X_0, Y_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0, Y_0))\rangle$, $\langle \overline{\mathrm{Apr}}(X_0, Y_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0, Y_0))\rangle$ and $\alpha(X_0, Y_0)$, and Algorithm 3 to learn $\langle \mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)\rangle$, $\langle \mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)\rangle$ and $\beta(B_0)$;
If $\underline{\mathrm{Apr}}(X_0, Y_0) \preccurlyeq \mathcal{H}_n(B_0) \preccurlyeq \overline{\mathrm{Apr}}(X_0, Y_0)$ or $\overline{\mathrm{Apr}}(B_0) \subseteq \mathcal{L}_n(X_0, Y_0) \subseteq \underline{\mathrm{Apr}}(B_0)$ does not hold
  Return "(X_0, Y_0) and B_0 are jointly concept-uninducible";
Else
  do
$$\langle (X_1, Y_1), B_1\rangle \leftarrow \langle \underline{\mathrm{Apr}}(X_0, Y_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0, Y_0))\rangle \bigwedge \langle \mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)\rangle,$$
$$\langle (X_2, Y_2), B_2\rangle \leftarrow \langle \overline{\mathrm{Apr}}(X_0, Y_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0, Y_0))\rangle \bigvee \langle \mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)\rangle,$$
$$\gamma((X_0, Y_0), B_0) \leftarrow min\{\alpha(X_0, Y_0), \beta(B_0)\};$$
End If
Return $\langle (X_1, Y_1), B_1\rangle$, $\langle (X_2, Y_2), B_2\rangle$ and $\gamma((X_0, Y_0), B_0)$.

Based on the time complexity of Algorithms 2 and 3, we know that the time complexity of Algorithm 4 is $O(|U_n||A_n|)$.

Example 8 (Continued with Example 7)**.** It has been known from Example 7 that $(X_0, Y_0) = (\{x_1, x_4, x_5, x_7, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}\})$ and $B_0 = \{A_{21}, A_{22}\}$ are jointly concept-inducible. Moreover, in Example 5, two approximate cognitive concepts $\langle(\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\}), \{A_{21}, A_{22}\}\rangle$ and $\langle(\{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\}), \{A_{21}, A_{22}, A_{23}\}\rangle$ were learnt from (X_0, Y_0) with the learning accuracy $\alpha(X_0, Y_0) = \frac{6}{7}$. Additionally, in Example 6, an exact cognitive concept $\langle(\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\}), \{A_{21}, A_{22}\}\rangle$ was learnt from B_0 with the learning accuracy $\beta(B_0) = 1$.

Then, based on Eqs. (21) and (22), we can learn two approximate three-way cognitive concepts $\langle(\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\}), \{A_{21}, A_{22}\}\rangle$ and $\langle(\{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\}), \{A_{21}, A_{22}, A_{23}\}\rangle$ from (X_0, Y_0) and B_0 with the learning accuracy $\gamma((X_0, Y_0), B_0) = \frac{6}{7}$. That is to say, (X_0, Y_0) and B_0 are not completely matched with each other, but they can induce two approximate cognitive concepts with 86% accuracy. Moreover, the following decisions can be made by the induced approximate cognitive concepts:

- the reviewers from Domains 1–3 accepted the manuscripts $x_1, x_4, x_5, x_7, x_8, x_{10}$, but rejected $x_2, x_3, x_6, x_9, x_{11}$ and x_{12};
- the reviewers from Domains 1 and 2 accepted the manuscripts x_1, x_4, x_5, x_{10}, but rejected x_2, x_3, x_6 and x_{11}.

5. Numerical experiments

In this section, we conduct some numerical experiments to evaluate the performance of the proposed learning methods.

In the experiments, we chose five datasets from UCI Machine Learning Repository [8]: the *Letter Recognition* dataset, *KEGG Metabolic Relation Network* dataset, *Skin Segmentation* dataset, *3D Road Network* dataset and *Poker Hand* dataset. The detailed information about these datasets is described in Table 3. In the experiments, the first attribute "Pathway text" in KEGG Metabolic Relation Network dataset was excluded since it is symbolic.

In order to generate standard datasets (i.e., their attributes are all Boolean), a data pre-processing technique was applied to the five chosen datasets. See Table 4 for the details, where "/" means "taking no action", "Bisection" means "splitting the

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Table 3
The detailed information about the five chosen datasets in the experiments.

Dataset	Instances	Attributes
Letter Recognition	20,000	16 (Discrete, each having 16 values)
KEGG Metabolic Relation Network	53,414	3 (Boolean), 20 (continuous)
Skin Segmentation	245,057	1 (Discrete, 2 values), 3 (continuous)
3D Road Network	434,874	4 (Continuous)
Poker Hand	1,025,010	11 (Discrete)

Table 4
Converting the five chosen datasets into standard datasets.

Dataset	Data pre-processing of attributes	Scaling
Letter Recognition	/	Nominal scale
KEGG Metabolic Relation Network	Bisection except the Boolean ones	Nominal scale
Skin Segmentation	Being divided into six equal segments except the discrete one	Nominal scale
3D Road Network	Being divided into six equal segments	Nominal scale
Poker Hand	Bisection	Nominal scale

Table 5
Designing three-way cognitive computing systems of the obtained standard datasets.

TWCCS	Design of parameters		
$\mathcal{F}^{(1)}$	$U_1=\{1\text{–}2000\}$, $A_1=\{1\text{–}25\}$,	$U_2=\{1\text{–}4000\}$, $A_2=\{1\text{–}50\}$,	$U_3=\{1\text{–}6000\}$, $A_3=\{1\text{–}75\}$,
	$U_4=\{1\text{–}8000\}$, $A_4=\{1\text{–}100\}$,	$U_5=\{1\text{–}10{,}000\}$, $A_5=\{1\text{–}125\}$,	$U_6=\{1\text{–}12{,}000\}$, $A_6=\{1\text{–}152\}$,
	$U_7=\{1\text{–}14{,}000\}$, $A_7=\{1\text{–}178\}$,	$U_8=\{1\text{–}16{,}000\}$, $A_8=\{1\text{–}204\}$,	$U_9=\{1\text{–}18{,}000\}$, $A_9=\{1\text{–}230\}$,
	$U_{10}=\{1\text{–}20{,}000\}$, $A_{10}=\{1\text{–}256\}$		
$\mathcal{F}^{(2)}$	$U_1=\{1\text{–}8902\}$, $A_1=\{1\text{–}15\}$,	$U_2=\{1\text{–}17{,}804\}$, $A_2=\{1\text{–}20\}$,	$U_3=\{1\text{–}26{,}706\}$, $A_3=\{1\text{–}25\}$,
	$U_4=\{1\text{–}35{,}608\}$, $A_4=\{1\text{–}30\}$,	$U_5=\{1\text{–}44{,}510\}$, $A_5=\{1\text{–}37\}$,	$U_6=\{1\text{–}53{,}414\}$, $A_6=\{1\text{–}43\}$
$\mathcal{F}^{(3)}$	$U_1=\{1\text{–}81{,}685\}$, $A_1=\{1\text{–}10\}$,	$U_2=\{1\text{–}163{,}370\}$, $A_2=\{1\text{–}15\}$,	$U_3=\{1\text{–}245{,}057\}$, $A_3=\{1\text{–}20\}$
$\mathcal{F}^{(4)}$	$U_1=\{1\text{–}144{,}958\}$, $A_1=\{1\text{–}12\}$,	$U_2=\{1\text{–}289{,}916\}$, $A_2=\{1\text{–}18\}$,	$U_3=\{1\text{–}434{,}874\}$, $A_3=\{1\text{–}24\}$
$\mathcal{F}^{(5)}$	$U_1=\{1\text{–}341{,}670\}$, $A_1=\{1\text{–}10\}$,	$U_2=\{1\text{–}683{,}340\}$, $A_2=\{1\text{–}15\}$,	$U_3=\{1\text{–}1{,}025{,}010\}$, $A_3=\{1\text{–}22\}$

values of each attribute, from small to large, into two disjoint intervals whose lengths are the same", and "Being divided into six equal segments" means "splitting the values of each attribute, from small to large, into six pairwise disjoint intervals whose lengths are the same". Moreover, the scaling approach [46] was used to transform them into standard datasets. Here, we denote the obtained standard datasets by Datasets 1–5 which are in fact formal contexts.

Furthermore, Datasets 1, 2, 3, 4 and 5 were divided into segments for designing their corresponding three-way cognitive computing systems: $\mathcal{F}^{(1)}=\bigcup_{i=2}^{10}\{\mathcal{F}^{(1)}_{\mathcal{H}_i\mathcal{L}_i}\}$, $\mathcal{F}^{(2)}=\bigcup_{i=2}^{6}\{\mathcal{F}^{(2)}_{\mathcal{H}_i\mathcal{L}_i}\}$, $\mathcal{F}^{(3)}=\bigcup_{i=2}^{3}\{\mathcal{F}^{(3)}_{\mathcal{H}_i\mathcal{L}_i}\}$, $\mathcal{F}^{(4)}=\bigcup_{i=2}^{3}\{\mathcal{F}^{(4)}_{\mathcal{H}_i\mathcal{L}_i}\}$ and $\mathcal{F}^{(5)}=\bigcup_{i=2}^{3}\{\mathcal{F}^{(5)}_{\mathcal{H}_i\mathcal{L}_i}\}$, respectively. See Table 5 for the details, where TWCCS is the abbreviation of "Three-way cognitive computing system". In the table, $U_i=\{p\text{–}q\}$ means that U_i is constituted by the objects between the pth and qth objects including the endpoints, so does A_i. In addition, we show how $\mathcal{Q}(A_{i-1})\preceq\mathcal{Q}(A_i)$ $(i=2,3,\ldots,10)$ were designed in $\mathcal{F}^{(1)}$. Specifically, $\mathcal{Q}(A_{i-1})=\{A_{(i-1)1},A_{(i-1)2},A_{(i-1)3},A_{(i-1)4},A_{(i-1)5},A_{(i-1)6}\}$, where the elements of each $A_{(i-1)j}$ were taken from A_{i-1} in sequence. The cardinality of $A_{(i-1)j}$ $(j=1,2,3,4,5)$ is $[\frac{|A_{i-1}|}{5}]$, while that of $A_{(i-1)6}$ is the remainder of $|A_{i-1}|$ divided by 5. Similarly, $\Delta A_{i-1}=A_i-A_{i-1}=\{\Delta A_{(i-1)1},\Delta A_{(i-1)2},\Delta A_{(i-1)3},\Delta A_{(i-1)4},\Delta A_{(i-1)5},\Delta A_{(i-1)6}\}$, where the elements of each $\Delta A_{(i-1)j}$ were taken from ΔA_{i-1} in sequence. The cardinality of $\Delta A_{(i-1)j}$ $(j=1,2,3,4,5)$ is $[\frac{|\Delta A_{i-1}|}{5}]$, while that of $\Delta A_{(i-1)6}$ is the remainder of $|\Delta A_{i-1}|$ divided by 5. Then $\mathcal{Q}(A_i)=\{A_{i1},A_{i2},A_{i3},A_{i4},A_{i5},A_{i6}\}$ was defined by taking $A_{ij}=A_{(i-1)j}\cup\Delta A_{(i-1)j}$. As a result, $\mathcal{Q}(A_{i-1})\preceq\mathcal{Q}(A_i)$ is satisfied. The cases of $\mathcal{F}^{(2)}$, $\mathcal{F}^{(3)}$, $\mathcal{F}^{(4)}$ and $\mathcal{F}^{(5)}$ were dealt with in a manner similar to $\mathcal{F}^{(1)}$, which is omitted here for convenience of presentation.

In the experiments, we took $\alpha=\frac{3}{4}$ and $\beta=\frac{1}{4}$. Notice that the above standard datasets are formal contexts with the input data being ones and zeros. Then the evaluation function $f_{B_i}(x)$ $(B_i\in 2^{\mathcal{Q}(A_i)})$ was set to be the ratio of the number of ones given to x to that of ones and zeros given to x under the columns $\cup B_i$. Moreover, in order to guarantee the successful implementation of sequential three-way decisions, the information on the objects which had been classified into positive or negative regions in last cognitive computing state, was omitted when it comes into next cognitive computing state.

Then, Algorithm 1 was applied to Datasets 1–5. The corresponding running time is reported in Table 6, where $|U|$ is the cardinality of object set, $|A|$ is that of attribute set, and n is the number of three-way cognitive computing states. It can be seen from Table 6 that Algorithm 1 is reasonably efficient even for the largest dataset.

Using Algorithm 1, we have obtained the three-way granular concepts $G^{(1)}_{\mathcal{L}_{10}\mathcal{H}_{10}}$, $G^{(2)}_{\mathcal{L}_6\mathcal{H}_6}$, $G^{(3)}_{\mathcal{L}_3\mathcal{H}_3}$, $G^{(4)}_{\mathcal{L}_3\mathcal{H}_3}$ and $G^{(5)}_{\mathcal{L}_3\mathcal{H}_3}$ of the three-way cognitive computing systems $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$, $\mathcal{F}^{(3)}$, $\mathcal{F}^{(4)}$ and $\mathcal{F}^{(5)}$. So, based on the theoretical results in Section 4, these granular concepts can be further used to learn three-way cognitive concepts from a given clue. Without loss of gener-

Table 6
Experimental results.

Dataset	$\lvert U \rvert$	$\lvert A \rvert$	n	Running time(s)			
				Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 4
Dataset 1	20,000	256	10	0.6843	0.0019	0.0021	0.0042
Dataset 2	53,414	43	6	1.1522	0.0022	0.0034	0.0064
Dataset 3	245,057	20	3	4.2865	0.0172	0.0123	0.2803
Dataset 4	434,874	24	3	12.8106	0.0180	0.0228	0.7907
Dataset 5	1,025,010	22	3	199.4306	0.0233	0.0297	0.8136

ality, we generated 100 clues randomly for Algorithm 2 as well as Algorithms 3 and 4. Then, Algorithms 2–4 were applied to Datasets 1–5. Here, Algorithms 2–4 were repeated 100 times since they are only able to achieve the learning task of a clue each time. The average running time of Algorithms 2–4 is also reported in Table 6. It can be observed from the table that they are all quite fast even for the largest dataset.

6. Final remarks

In this section, we give some remarks to conclude the paper.

(i) A brief summary of our work. To uncover the essential idea of three-way concepts for solving decision-making problems, we have discussed three-way cognitive concept learning via multi-granularity. Specifically, an axiomatic method of forming three-way cognitive concepts has firstly been proposed based on multi-granularity and three-way-decision-making principles. Then, a three-way cognitive computing system has been designed for learning composite three-way granular concepts. Moreover, cognitive processes have been simulated by the idea of low and upper approximations to learn three-way cognitive concepts from a given clue. Finally, numerical experiments have been conducted to evaluate the performance of the proposed learning methods.

(ii) The significance of our research. It is noticed that many different types of three-way concepts have been proposed in the existing literature and each of them has different properties. It is essential to identify which properties are intrinsic for characterizing three-way concepts in order to understand the basic decision-making mechanism of three-way concepts. Using multi-granularity and three-way-decision-making principles, our research has successfully clarified three properties which can be jointly used as axioms to characterize three-way concepts. In addition, as discussed in Section 2.2, these intrinsic properties have explicit semantics.

(iii) The advantages of our methods. We have designed a three-way cognitive computing system to learn granular concepts and proposed concept learning methods for simulating cognitive processes. Our three-way cognitive computing system can update three-way granular concepts as objects and attributes increase. What is more, the proposed concept learning methods can help to remember three-way cognitive concepts from a given clue. Besides, as shown by the experiments conducted in Section 5, our learning methods are quite efficient; they only take less than 200 seconds for the dataset with more than one million instances. Therefore, it seems possible for our methods to be applied in big data if some parallel computing techniques could be successfully developed.

(iv) The differences and similarities between our study and the existing ones. The idea of sequential three-way decisions has been adopted in this paper to establish an axiomatic method of forming three-way concepts. Granular computing has been incorporated into three-way cognitive concepts for constructing information granules, which guarantees that three-way granular concepts can be defined and used to remember new cognitive concepts from a given clue. What is more, the proposed three-way cognitive operators $\mathcal{H}$ and $\mathcal{L}$ form an isotone Galois connection between $2^{\mathcal{Q}(A)}$ and $\mathcal{T}(U)$. So, they are completely different from the classical cognitive operators [16] which form an antitone Galois connection between 2^A and 2^U. In other words, these two kinds of cognitive operators have different cognitive mechanisms.

Nevertheless, there are some similarities between our study and the existing ones. For instance, multi-granularity has been designed to be monotonous for supporting sequential three-way decisions, which was realized by $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$ in the process of information updating. As usual, our sequential three-way decisions also become more and more effective from last three-way cognitive computing state to the next one. If information can be updated continually, the final result of our sequential three-way decisions will degenerate into two-way decisions (i.e., boundary regions disappear).

(v) An outlook for further study. Note that the classical cognitive operators have been reconsidered to fit the big data environment [14]. As a matter of fact, such a problem is also encountered in three-way cognitive operators. So, it is still necessary to redesign three-way cognitive operators for meeting different requirements of big data such as large-scale, multi-source and heterogeneous data. Moreover, in our opinion, cognitive logic should be introduced into three-way cognitive computing system for effectively simulating the human brain behaviors including learning, reasoning and so on. These issues will be investigated in our future work.

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