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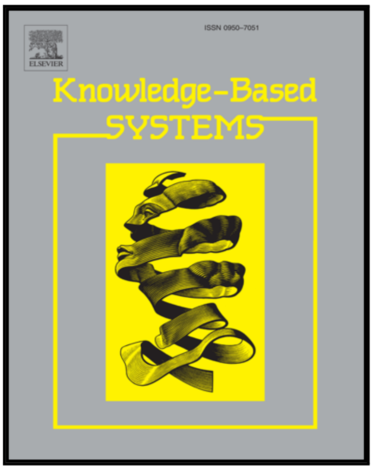

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Multigranulation fuzzy rough set over two universes and its application to decision making

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Abstract

The original Pawlak's rough set approach based on indiscernibility relation (single granularity) has been extended to multigranulation rough set structure in the recent years. Multigranulation rough set approach has became a flouring research direction in rough set theory. This paper considers rough approximation of a fuzzy concept under the framework of multigranulation over two different universes of discourse, i.e., multigranulation fuzzy rough set models over two universes. We present three types of multigranulation fuzzy rough set over two universes by the constructive approach, respectively. Some interesting properties of the proposed models are discussed and also the interrelationships between the proposed models and the existing rough set models are given. We then propose a new approach to a kind of multiple criteria group decision making problem based on multigranulation fuzzy rough set model over two universes. The decision rules and algorithm of the proposed method are given and an example of handling multiple criteria group decision making problem of clothes ranking illustrates this approach. The main contribution of this paper is twofold. One is to establish the multigranulation fuzzy rough set theory over two universes. Another is to try presenting a new approach to multiple criteria group decision making based on multigranulation fuzzy rough set over two universes. The proposed models not only enrich the theory of multigranulation rough set but also make a tentative to provide a new perspective for multiple criteria group decision making with uncertainty.

Keywords: Rough set, Rough fuzzy set, Rough set over two universes, Multigranulation fuzzy rough set, Multiple criteria group decision making.

1. Introduction

Recent changes in philosophical ideas (mechanistic reductionism to evolutionist holism), methodology (from the search for truth to the search for knowledge), and technology (communication networks) that have occurred in the knowledge society have led to use of more open and flexible scientific approaches to multiactor decision making [32]. These new approaches have to encompass the integration of intangible and subjective aspects associated with the human (subjective) factor in the resolution of problems [2]. Several approaches with the new emerging mathematical theories, tools, and applications have been developed in the fields of management science, operational research, and industrial engineering [3, 9, 10, 16, 58, 65]. Rough set theory [35] is a new mathematical tool for studying intelligent decision systems characterized by insufficient and incomplete information. So far rough set theory has became one of important and effective methods for decision making problem with incomplete and inaccurate available information [6, 7, 11 – 13, 24, 27, 34, 42, 51]. Also, it has established an effective qualitative method for dealing with uncertainty in a wide variety of applications related to knowledge discovery and pattern recognition [17, 30, 46, 62 – 64, 57, 67, 71, 72], etc.

Pawlak rough set deals with the approximation of a subset of universe by the lower and upper approximations based on an indiscernibility relation determined by attributes. By using the concept of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. As is well known, indiscernibility relation on universe is the key concept in constructing on the lower and upper approximations of an approximated set using the Pawlak rough set theory. However, the indiscernibility relation on universe may be incapable to many practical applications, particularly in handling real-valued, symbolic attribute values and hybrid attribute values. Facing these problems in application, several important and interesting generalizations of Pawlak rough set model have been established in the past few years, such as fuzzy rough set model [31, 43, 65], general binary relation based rough set model [41, 44], neighborhood rough set model, probabilistic rough set [59], rough set over two universes [36, 44, 58], decision-theoretic rough sets (DTRS) [60], dominance-based rough sets approach (DRAS) [11, 45, 46], composite rough sets (CRS) [68], rough soft set and soft rough set [70], and etc.

A common characteristic of both Pawlak rough set model and all extension models is that the approximated set is usually described by one single granularity (such as an indiscernibility relation, tolerance relation, dominance relation or compatible relation) on universe. From the point of view of granular computing, an indiscernibility relation on universe can be regarded as a granularity, and then a partition (or a covering) of the universe can be regarded as a granulation space. As pointed out in [61], the single granular (or binary relation) on universe used in the existing rough set models provide an efficient approach to deal with the uncertainty of decision-making problem and also limits the application of the models. For instance [8, 28, 29, 33], in the practice of comprehensive evaluation group decision making, the decision-makers often need to acquire the evaluation results of all objects of universe with respect to different evaluation indices based on their personal preference. Then different decision-makers may select different evaluation indices as the optimal combination to express their preference evaluation. So, the preference evaluations related to the optimal combination of the selected evaluation indices given by different decision-makers are made of a multiple granularity structure of all objects of universe for the same decision-making problem. For that reason, the existing rough set models based on one single binary relation are incapable of handling this type of decision problem.

To more widely apply the rough set theory in practical applications, Qian et al. [37 − 39] extended Pawlak's single granularity rough set model to multigranulation rough set model. So far multigranulation rough set [39] has become an attractive topic in artificial intelligence and management science and has attracted a broad range of studies from both theoretical and application aspects [1, 4, 14, 18, 20−23, 25, 26, 40, 54−56, 61, 66, 69]. Detailed review about the existing studying of multigranulation rough set theory is suggested to refer to Ref. [28, 29, 42]. Now multigranulation rough set approach has proved to provide a new kind of information fusion strategy compared to the single granularity rough sets. With regard to some special information systems, such as multi-source information systems, distributive information systems, groups of intelligent agents and multiple attribute group decision making, the existing single granularity rough sets can not be used to deal with data from these information systems and uncertainty decision making, but multigranulation rough set can.

Though there has been many researches about various generalized multigranulation rough set models, there has less effort to study the multigranulation rough set over two universes. Moreover, many of the existing multigranulation rough set models are focused on rough approximation of a crisp concept but less effort to the study of rough approximation of a fuzzy concept of universe. Meanwhile, some uncertainty decision making problems could not be solved by the existing single granularity-based rough set approaches over two universes. Let us consider an example of multiple criteria group decision making problem in management science: the decision making of medical diagnosis in clinics [39, 42]. In general, a symptom is an uncertainty index of whether a disease may occur or not. Given a specific patient in clinic, the patient may show many symptoms, just as each disease could have many basic symptoms. Symptoms and diseases belong to two different universes, although they are interrelated with each other. Thus, uncertainty arises when describing the interrelations between symptoms and diseases in clinic. Therefore, two or more different universes are needed when expressing the decision making problem of the medical diagnosis in clinics. At the same time, the symptoms of the patient are usually described by a fuzzy set on the symptom set in practice. Moreover, suppose there is a critically ill patient in emergency department, in order to make a exactly diagnosis according to the symptoms for the critically ill patient, the doctor could invite multiple related department experts such as the surgeon, the physician and the urologist to make a consultation and then gives a reasonable diagnosis decision making. As is well known, the surgeon, the physician and the urologist will present their opinions for the critically ill patient that which disease the patient has based on the domain knowledge of themselves in the process of consultation. So, rough approximation of a fuzzy concept by multiple granularity under the framework of two different universes could provide a new way to handle the aforementioned uncertainty decision making problem in realty.

With reference to the requirement of the applications in practice, as well as the complement of the theoretical aspect of multigranulation rough set, this paper mainly focuses on rough approximation of a fuzzy concept in multigranulation (fuzzy) approximation space over two universes, i.e., multigranulation rough fuzzy set (fuzzy rough set) models over two universes. We will define three types of multigranulation rough fuzzy set (fuzzy rough set) based on a family of arbitrary (fuzzy) binary relation over two universes, respectively. Moreover, we try to make an attempt to build up a general framework of the decision methodology based on multigranulation fuzzy rough set theory over two universes.

The rest of this paper is organized as follows. Section 2 reviews some basic concepts used in the following sections. In Section 3, we investigate the theory of multigranulation rough fuzzy set over two universes. Section 4 presents fuzzy rough set on multigranulation fuzzy approximation space over two universes. In Section 5, we present an application of the multigranulation fuzzy rough set approach to multiple criteria group decision making problem. At last, we conclude our research and set out further research directions in Section 6.

2. Preliminaries

2.1. Fuzzy set and Pawlak rough set

Let U be a non-empty finite universe. A fuzzy set of universe U is defined by the mapping $A(\bullet) : U \longrightarrow [0, 1]$, where the $A(x)$ denotes the membership of the element $x (x \in U)$ with the fuzzy set A. We use $F(U)$ stand for all fuzzy subsets of universe U.

For any $A \in F(U)$, the r level set and strong r level set of A will be denoted by A_r and A_{r+}, respectively. That is, $A_r = \{x \in U | A(x) \geq r\}$ and $A_{r+} = \{x \in U | A(x) > r\}$, where $r \in [0, 1]$, the unit interval, $A_0 = U$ and $A_{1+} = \emptyset$.

Next, we introduce the concept of Pawlak rough set [36].

Let U be a non-empty finite universe and R be an equivalence relation of $U \times U$. The equivalence relation R induces a partition of U, denoted by $[x]_R$ or $[x]$, and $U/R = \{[x]|x \in U\}$ stands for the equivalence classes of x. Then (U, R) is the Pawlak approximation space.

Let (U, R) be the Pawlak approximation space. For any $X \subseteq U$, the lower and upper approximations of X with respect to (U, R) are defined as follows, respectively.

$\underline{R}(X) = \{x \in U|[x] \subseteq X\}$,

$\overline{R}(X) = \{x \in U|[x] \cap X \neq \emptyset\}$.

The lower approximation $\underline{R}X$ is the union of all elementary sets which are the subset of X, and the upper approximation $\overline{R}X$ is the union of all elementary sets which have a non-empty intersection with X.

2.2. Rough set model over two universes

Next, we review the basic concepts of rough set model on two universes. A detailed description of the model can be seen in Wong et al. [52], Yao et al. [58] and Pei and Xu [36, 44].

Definition 2.1 Let U and V be two universes, and R be a binary relation from U to V, i.e. a subset of $U \times V$. R is said to be compatible, or a compatibility relation, if, for any $u \in U$; $v \in V$, there exist $t \in V$ and $s \in U$ such that $(u, t), (s, v) \in R$.

Definition 2.2 Let U, V be two universes, R be a compatibility relation from U to V. The mapping $F : U \to 2^V, u \mapsto \{v \in V|(u, v) \in R\}$ is called the mapping induced by R.

Obviously, the above-defined binary relation R can uniquely determine the mapping F, and vice versa. Then the rough set over two universes is defined as follows:

Let U and V be two universes, and R be a compatibility relation from U to V. The ordered triple (U, V, R) is called a (two-universe) approximation space. The lower and upper approximations of $Y \subseteq V$ are, respectively, defined as follows:

$\underline{Apr}_F(Y) = \{x \in U|F(x) \subseteq Y\}$,

$\overline{Apr}_F(Y) = \{x \in U|F(x) \cap Y \neq \emptyset\}$.

The ordered set-pair $(\underline{Apr}_F(Y), \overline{Apr}_F(Y))$ is called a generalized rough set, and the ordered operator-pair $(\underline{Apr}_F, \overline{Apr}_F)$ is an interval structure. Particularly, Y is called definable with (U, V, R) if $\underline{Apr}_F(Y) = \overline{Apr}_F(Y)$. Otherwise, Y is an indefinable set. Meanwhile, the model presented above is called rough set over two universes.

The set $\underline{Apr}_F(Y)$ consists of elements of U which are only compatible with those elements in Y, while the set $\overline{Apr}_F(Y)$ consists of elements of U which are compatible with at least one element in Y. Therefore, the former can be interpreted as the pessimistic description and the latter as the optimistic description of Y.

2.3. Multigranulation rough set model

Definition 2.3 [37] Let $K = (U, R)$ be a knowledge base and P, Q be two equivalence relations of universe U. For any $X \subseteq U$, the lower and upper approximations of X with respect to P and Q are defined as follows, respectively.

$\underline{X}_{P+Q} = \{x \in U|[x]_P \subseteq X \vee [x]_Q \subseteq X\}$,

$\overline{X}^{P+Q} = (\underline{X^c}_{P+Q})^c$ (where X^c stands for the complementary of X).

Generally speaking, we call X definable with respect to equivalence relation P and Q if $\underline{X}_{P+Q} = \overline{X}^{P+Q}$; Otherwise, X is a rough set with respect to $K = (U, R)$. Moreover, it is easy to know that any one equivalence relation will form a partition of universe U , i.e., any one equivalence relation of the universe forms a granularity structure of the universe of discourse. So, we call X the multigranulation rough set when $\underline{X}_{P+Q} \neq \overline{X}^{P+Q}$ because the granularity structure of the universe U is generated by two different equivalence relations P and Q.

Definition 2.4 [42] Let U, V be two non-empty finite universes. $\mathfrak{R}$ is a family binary compatibility relation between U and V induced by binary mapping family $F_i : U \to 2^V, u \mapsto \{v \in V|(u, v) \in R_i\}$, $R_i \in \mathfrak{R}, i = 1, 2, \cdots, m$. We call triple ordered set $(U, V, \mathfrak{R})$ the multigranulation approximation space over two universes.

It can be easily seen that the multigranulation approximation space $(U, V, \mathfrak{R})$ will degenerate into the approximation space over two universes (U, V, R) if there exists only one binary mapping F on universe U and V. So, the concept of multigranulation approximation space is a natural generalization of the approximation space over two universes.

Definition 2.5 [50] We call fuzzy subset $R \in F(U \times V)$ the binary fuzzy relation from U to V. $R(x, y)$ is the related degree between elements x and y, where $(x, y) \in U \times V$. Particularly, for any $x \in U$, R is called serial fuzzy binary relation from U to V if there exists element $y \in V$ and satisfies $R(x, y) = 1$.

Definition 2.6 Let U, V be two non-empty finite universes. R is a family binary fuzzy relation between U and V, $R_i \in F(U \times V)$ and $R_i \in \mathrm{R}, i = 1, 2, \cdots, m$. We call triple ordered set (U, V, R) the multigranulation fuzzy approximation space over two universes.

3. Multigranulation rough fuzzy set over two universes

In this section, we will systematically discuss the rough approximation of a fuzzy concept with respect to multiple granularity between two different universes. Under the framework of two universes, we will present the optimistic multigranulation rough set, pessimistic multigranulation rough set and α variable precision multigranulation rough set over multigranulation (fuzzy) approximation space over two universes, respectively.

This section considers rough approximation of a fuzzy concept with respect to multigranulation approximation space. We will give three different multigranulation rough fuzzy set models over approximation space based on the corresponding decision making background of management science, respectively.

3.1. Optimistic multigranulation rough fuzzy set over two universes

The idea of optimistic multigranulation rough set reflects the decision making of risk preferring decision-maker in practice of management science [42]. Generally speaking, in the practice of decision making of management science, there are many non-determined decision making problems due to the difficult structure of the decision making problem itself, the complexity of the decision making environment and the inaccuracy and incompleteness available information. Also, different patterns of decision making occur because of the different risk preferences of decision-makers. For instance, consider a decision making problem involving a portfolio investment with a fuzzy decision making object. In this case, risk preferring decision-makers may select an investment project that does not satisfy the pre-determined investment criteria with respect to all features of the project. The model of optimistic multigranulation rough fuzzy set over two universes can depict this kind of non-determined decision problem with a fuzzy decision making object. Meanwhile, the lower and upper approximations of this model can be interpreted as a kind of generalized risk decision rule in traditional risk decision making with uncertainty [42, 46].

Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$. The optimistic lower approximation $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ and optimistic upper approximation $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ of fuzzy set A in $(U, V, \mathfrak{R})$ are defined as follows, respectively.

$\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) = \min\{A(y)|y \in \bigvee_{i=1}^{m} F_i(x), y \in V\}, \quad x \in U;$

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) = \max\{A(y)|y \in \bigvee_{i=1}^{m} F_i(x), y \in V\}, \quad x \in U.$

Where $\bigvee_{i=1}^{m} F_i(x) = F_1(x) \vee F_2(x) \vee \cdots \vee F_m(x)$ and then $y \in F_1(x) \vee F_2(x)$ means that $y \in F_1(x)$ or $y \in F_2(x)$. Meanwhile, the minimum and maximum become inf and sup when universe U and V are infinite set.

It is easy to know that $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ and $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ are two fuzzy sets of universe U. Furthermore, A is called a definable fuzzy set on multigranulation approximation space over two universes $(U, V, \mathfrak{R})$ when $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) = \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$. Otherwise, we call the set-pair $(\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))$ the optimistic multigranulation rough fuzzy set over two universes.

In fact, from the point of view of risk decision making with uncertainty, the lower approximation $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ can be regarded as the " min − max " rule and the upper approximation $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ can be regarded as the " max − max " rule. So, the upper approximation of any fuzzy set with respect to multigranulation approximation space over two universes is the type of optimistic decision model in traditional risk decision making with uncertainty.

Remark 3.1 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$. Then

(1) If $U = V$, then $R_i \in \mathfrak{R}, i = 1, 2, \cdots, m$ degenerates into arbitrary binary relation induced by binary mapping $F_i(i = 1, 2, \cdots, m)$ of universe U. Then optimistic multigranulation rough fuzzy set over two universes $(\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))$ degenerates into optimistic multigranulation rough fuzzy set on single universe [58].

(2) If $R_1 = R_2 = \cdots = R_m$, there are

$\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) = \min\{A(y)|y \in F_i(x), i = 1, 2, \cdots, m, x \in U, y \in V\} = \underline{R_i}(A),$

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) = \max\{A(y)|y \in F_i(x), i = 1, 2, \cdots, m, x \in U, y \in V\} = \overline{R_i}(A).$

Then, $(\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))$ degenerates into rough fuzzy set over two universes [37, 38]. Furthermore, $(\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))$ degenerates into rough fuzzy set based on general binary relation over the single universe when $U = V$ [42].

(3) If A is a crisp set of the universe V, there is $A(y) = 0$ or $A(y) = 1$ for any $y \in V$, then

$\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) = \{x \in U|F_1(x) \subseteq A \vee F_2(x) \subseteq A \vee \cdots \vee F_m(x) \subseteq A\},$

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) = \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A^c)^c = \{x \in U|F_1(x) \cap A \neq \varnothing \wedge F_2(x) \cap A \neq \varnothing \wedge \cdots \wedge F_m(x) \cap A \neq \varnothing\}.$

Therefore, the $(\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))$ degenerates into the multigranulation rough set over two universes [41]. Furthermore, the $(\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))$ degenerates into the multigranulation rough set on the single universe [31] when $U = V$ and the general binary relation-based rough set of the single universe [35] when there are $U = V$ and $R_1 = R_2 = \cdots = R_m$.

It can be easily seen that multigranulation rough fuzzy set over two universes is a combination of the existing rough fuzzy set [53] and multigranulation rough set over two universes [42]. That is, both the rough fuzzy set and multigranulation rough set over two universes are the special case of multigranulation rough fuzzy set over two universes. So, the multigranulation rough fuzzy set over two universes could deal with more complexity decision making problems with inaccuracy and incompleteness available information in management science. Meanwhile, we have the following results.

Proposition 3.1 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$ and $x \in U$, there are

(1) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) \leq \bigvee_{i=1}^{m} \underline{R_i}(A)(x),$

(2) $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) \geq \bigvee_{i=1}^{m} \overline{R_i}(A)(x).$

Proof. (1) For any $x \in U$ and $y \in V$, there is $F_i(x) \subseteq \bigvee_{i=1}^{m} F_i(x)$. Then we have $\underline{R_i}(A)(x) = \min\{A(y)|y \in F_i(x)\} \geq \min\{A(y)|y \in \vee_{i=1}^{m} F_i(x)\} = \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x)$.

So, $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) \leq \bigvee_{i=1}^{m} \underline{R_i}(A)(x)$ holds.

(2) For any $x \in U$ and $y \in V$, because $F_i(x) \subseteq \bigvee_{i=1}^{m} F_i(x)$, we have $\overline{R_i}(A)(x) = \max\{A(y)|y \in F_i(x)\} \leq \max\{A(y)|y \in \vee_{i=1}^{m} F_i(x)\} = \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x)$.

So, $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) \geq \bigvee_{i=1}^{m} \overline{R_i}(A)(x)$ holds.

As is well known, the concept of the level set of a fuzzy set provides an effective method to transform a fuzzy set into a crisp set. At the same time, the decomposition theorem of Zadeh fuzzy set theory [53] gives the method how to construct a fuzzy set by using a family of crisp set on universe. In the following, we investigate the rough approximation of the level set of a fuzzy concept about multigranulation approximation space over two universes, and then present another way to construct the optimistic multigranulation rough fuzzy set over two universes.

Definition 3.1 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$, the lower and upper approximations of $r(r \in [0, 1])$ level set of A are defined as follows, respectively.

$\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_r) = \{x \in U | F_1(x) \subseteq A_r \vee F_2(x) \subseteq A_r \vee \cdots \vee F_m(x) \subseteq A_r\}$,

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_r) = \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_r^c)^c$
$= \{x \in U | F_1(x) \cap A_r \neq \emptyset \wedge F_2(x) \cap A_r \neq \emptyset \wedge \cdots \wedge F_m(x) \cap A_r \neq \emptyset\}$.

It is easy to know that both $\{\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_r) | r \in [0,1]\}$ and $\{\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_r) | r \in [0,1]\}$ are binary nest set over universe U with any fuzzy set A on universe V. Then we can obtain two new fuzzy sets over universe U based on this two binary nest sets by using the decomposition theorem of Zadeh fuzzy set theory [53] as follows:

$\underline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x) = \vee\{r | x \in \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_r)\}$
$= \vee\{r | F_1(x) \subseteq A_r \vee \cdots \vee F_m(x) \subseteq A_r\}$,

$\overline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x) = \vee\{r | x \in \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_r)\}$
$= \vee\{r | F_1(x) \cap A_r \neq \emptyset \wedge F_2(x) \cap A_r \neq \emptyset \wedge \cdots \wedge F_m(x) \cap A_r \neq \emptyset\}$.

Theorem 3.1 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$ and $x \in U$, there are

(1) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) = \underline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x)$,

(2) $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) = \overline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x)$.

Proof. (1) Suppose $r_0 = \underline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x)$ for any $x \in U$. Then there exists $F_i(i = 1, 2, \cdots, m)$ satisfies $F_i(x) \subseteq A_{r_0}$. This implies that $A(y) \geq r_0$ for any $y \in V$ and $\bigwedge_{y \in F_i(x)} A(y) \geq r_0$. Furthermore, there is $\min\{A(y) | y \in \vee_{i=1}^m F_i(x), y \in V\} \geq r_0$, i.e., $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) \geq r_0$.

So, we prove $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) \geq \underline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x)$.

On the other hand, suppose $r_0 = \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x)$ for any $x \in U$. There exists $F_i(i = 1, 2, \cdots, m)$ satisfies $\min\{A(y) | y \in \vee_{i=1}^m F_i(x), y \in V\} = r_0$. This implies $A(y) \geq r_0$ for any $y \in V$, i.e., $y \in F_i(x) \subseteq A_{r_0}(i = 1, 2, \cdots, m)$. By the definition of $\underline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)$, there is $\underline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x) \geq r_0$.

So, we prove $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) \leq \underline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x)$.

Hence, we prove that $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) = \underline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x)$ for any $x \in U$ holds.

It is similar to prove that $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) = \overline{\mathfrak{R}}'^{O}_{\sum_{i=1}^m R_i}(A)(x)$ holds.

Theorem 3.1 not only presents another way to construct the lower and upper approximations for optimistic multigranulation rough fuzzy set but also interprets the relationship with the optimistic multigranulation rough set over two universes [42].

We then investigate the relationship between the level set of a fuzzy concept and its lower and upper approximations with respect to multigranulation approximation space over two universes.

Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$, $0 < \beta, \alpha \leq 1$. We define

$\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\alpha = \{x \in U | \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) \geq \alpha\}$, and

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\beta = \{x \in U | \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) \geq \beta\}$

the α level set and β level set of the lower and upper approximations of A with respect to $(U, V, \mathfrak{R})$, respectively. Here the way of selection the thresholds α and β is same to the method given by Wu and Zhang [53].

By this definition, the relationship between the level set of a fuzzy set and its lower and upper approximations about multigranulation approximation space over two universes are as follows.

Proposition 3.2 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$, $0 < \beta \leq \alpha \leq 1$. We have

(1) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\alpha \supseteq \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_\alpha)$;

(2) $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\beta \supseteq \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_\beta)$;

(3) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_\alpha) \subseteq \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_\alpha)$;

(4) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\alpha \subseteq \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\beta$.

Proof. (1) For any $x \in U$, and $x \in \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_\alpha)$, there exists $F_i(i = 1, 2, \cdots, m)$ and satisfies $\bigvee_{i=1}^m F_i(x) \subseteq A_\alpha$. Then, for any $y \in V$, there is $y \in \bigvee_{i=1}^m F_i(x) \subseteq A_\alpha$, i.e., $A(y) \geq \alpha$. This implies $\min A(y) \geq \alpha$ for any $y \in \bigvee_{i=1}^m F_i(x)$. So, $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x) \geq \alpha$, i.e., $x \in \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\alpha$.

This proves $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\alpha \supseteq \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A_\alpha)$.

(2) can be proved with the similar way of (1).

(3) and (4) can be derived directly by the definitions.

In what follows, we present the approximate precision of optimistic multigranulation rough fuzzy set over two universes.

Definition 3.2 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$, $0 < \beta \leq \alpha \leq 1$. Then the accuracy and roughness of A about multigranulation approximation space over two universes are as follows:

$$\rho_A^O(\alpha, \beta) = \frac{|\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\alpha|}{|\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\beta|},$$

$$\sigma_A^O(\alpha, \beta) = 1 - \frac{|\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\alpha|}{|\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\beta|} = 1 - \rho_A^O(\alpha, \beta).$$

In particular, we convention that $\rho_A^O(\alpha, \beta) = 1$ when there is $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\beta = \emptyset$.

It is easy to know the method of defining the accuracy of any target set in multigranulation approximation space over two universes is similar to the general relation-based rough fuzzy set [42]. Furthermore, the following properties are clear.

Proposition 3.3 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$ and $x \in U$, $0 < \beta \leq \alpha \leq 1$. Then

(1) $0 \leq \rho_A^O(\alpha, \beta) \leq 1$, $0 \leq \sigma_A^O(\alpha, \beta) \leq 1$;

(2) $\rho_A^O(\alpha, \beta)$ is non-increasing for α and non-decreasing for parameter β;

(3) $\sigma_A^O(\alpha, \beta)$ is non-decreasing for α and non-increasing for parameter β.

Proposition 3.4 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A, B \in F(V)$ and $x \in U$, $0 < \beta \leq \alpha \leq 1$. Then

(1) $\rho_{A \cup B}^O(\alpha, \beta) |\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)_\beta \cup \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(B)_\beta| \geq$

$\rho_A^O(\alpha,\beta)|\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)_\beta| + \rho_B^O|\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(B)_\beta| - \rho^O_{A\cap B}(\alpha,\beta)$
$|\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)_\beta \cap \overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(B)_\beta|$,

(2) $\sigma^O_{A\cup B}(\alpha,\beta)|\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)_\beta \cup \overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(B)_\beta| \leq$
$\sigma_A^O(\alpha,\beta)|\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)_\beta| + \sigma_B^O|\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(B)_\beta| - \sigma^O_{A\cap B}(\alpha,\beta)$
$|\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)_\beta \cap \overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(B)_\beta|$.

Proof. The proof are similar to the Ref. [36].

Proposition 3.4 reveals the relationship between the intersection and union of the accuracy and roughness of any two fuzzy concepts with respect to multigranulation approximation space over two universes. In order to verify the validity of lower and upper approximations of optimistic multigranulation rough fuzzy set over two universes, consider a numerical example as follows.

Example 3.1 Let $U = \{x_1, x_2, x_3\}$ and $V = \{y_1, y_2, y_3\}$ be two non-empty universes. $R_i(i = 1, 2, 3)$ are binary relations induced by binary compatibility relations $F_i : U \mapsto V(i = 1, 2, 3)$. The elements of $R_i(i = 1, 2, 3)$ are given as follows, respectively.

$F_1(x_1) = \{y_1, y_2\}$, $F_1(x_2) = \{y_2\}$, $F_1(x_3) = \{y_1, y_3\}$,
$F_2(x_1) = \{y_2\}$, $F_2(x_2) = \{y_3\}$, $F_2(x_3) = \{y_1\}$;
$F_3(x_1) = \{y_2, y_3\}$, $F_3(x_2) = \{y_3\}$, $F_3(x_3) = \{y_1, y_3\}$.

Suppose the fuzzy set A on universe V as:

$$A = \frac{0.4}{y_1} + \frac{0.6}{y_2} + \frac{0.9}{y_3}.$$

We calculate the optimistic multigranulation lower and upper approximations of A as follows:

$\underline{\mathfrak{R}}^O_{\sum_{i=1}^3 R_i}(A)(x_1) = \min\{A(y_1), A(y_2), A(y_3)\}$
$= \{0.4, 0.6, 0.9\} = 0.4$,
$\overline{\mathfrak{R}}^O_{\sum_{i=1}^3 R_i}(A)(x_1) = \max\{A(y_1), A(y_2), A(y_3)\}$
$= \{0.4, 0.6, 0.9\} = 0.9$.

Similarly, we can calculate that

$\underline{\mathfrak{R}}^O_{\sum_{i=1}^3 R_i}(A)(x_2) = 0.6$, $\overline{\mathfrak{R}}^O_{\sum_{i=1}^3 R_i}(A)(x_2) = 0.9$.
$\underline{\mathfrak{R}}^O_{\sum_{i=1}^3 R_i}(A)(x_3) = 0.4$, $\overline{\mathfrak{R}}^O_{\sum_{i=1}^3 R_i}(A)(x_3) = 0.9$.

So, there are

$$\underline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A) = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3}$$

and

$$\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A) = \frac{0.9}{x_1} + \frac{0.9}{x_2} + \frac{0.9}{x_3}.$$

Taking $\alpha = 0.6$ and $\beta = 0.5$. We have

$$\rho_A^O(0.6, 0.5) = \frac{|\underline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)_{0.6}|}{|\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)_{0.5}|} = \frac{1}{3},$$

$$\sigma_A^O(0.6, 0.5) = \frac{2}{3}.$$

From Definition 2.4, for any two non-empty finite universes U and V, $\mathfrak{R}$ is a family binary compatibility relation between U and V induced by binary mapping family $F_i : U \to 2^V, u \mapsto \{v \in V|(u, v) \in R_i\}$, $R_i \in \mathfrak{R}, i = 1, 2, \cdots, m$. Then $\bigcup_{i=1}^m F_i$ and $\bigcap_{i=1}^m F_i$ $(i = 1, 2, \cdots, m)$ also are binary mappings between U and V, i.e., both $\bigcup_{i=1}^m F_i$ and $\bigcap_{i=1}^m F_i : U \to 2^V, u \mapsto \{v \in V|(u, v) \in \bigcup_{i=1}^m R_i\}$, and vice versa.

In fact, it is easy to know that the operations of $\bigcup_{i=1}^m F_i$ and $\bigcap_{i=1}^m F_i$ define two new binary relation between universe U and V, respectively. Meanwhile, the $\bigvee_{i=1}^m F_i$ means that the union operation of m binary relations between universe U and V. That is to say, the operations of $\bigcup_{i=1}^m F_i$ and $\bigcap_{i=1}^m F_i$ represent two new subsets of universe V, respectively. The $\bigvee_{i=1}^m F_i$ represents a family of subsets on the universe V. We use an example to illustrate the differences between $\bigvee_{i=1}^m F_i(x)$ and $\bigcup_{i=1}^m F_i(x)$ in the follows.

Example 3.2 (Continued from Example 3.1) According to the Example 3.1, we have the following results:

$\bigcup_{i=1}^m F_i(x_1) = F_1(x_1) \cup F_2(x_1) \cup F_3(x_1) = \{y_1, y_2, y_3\}$,
$\bigcup_{i=1}^m F_i(x_2) = F_1(x_2) \cup F_2(x_2) \cup F_3(x_2) = \{y_2, y_3\}$,
$\bigcup_{i=1}^m F_i(x_3) = F_1(x_3) \cup F_2(x_3) \cup F_3(x_3) = \{y_1, y_3\}$.

At the same time, we have

$\bigvee_{i=1}^m F_i(x_1) = F_1(x_1) \vee F_2(x_1) \vee F_3(x_1)$
$= \{y_1, y_2\} \vee \{y_2\} \vee \{y_2, y_3\}$,
$\bigvee_{i=1}^m F_i(x_2) = F_1(x_2) \vee F_2(x_2) \vee F_3(x_2) = \{y_2\} \vee \{y_3\} \vee \{y_3\}$,
$\bigvee_{i=1}^m F_i(x_3) = F_1(x_3) \vee F_2(x_3) \vee F_3(x_3)$
$= \{y_1, y_3\} \vee \{y_1\} \vee \{y_1, y_3\}$.

That is, there are

$\bigcup_{i=1}^m F_i(x_1) = \{y_1, y_2, y_3\}$,
$\bigcup_{i=1}^m F_i(x_2) = \{y_2, y_3\}$,
$\bigcup_{i=1}^m F_i(x_3) = \{y_1, y_3\}$,

and

$\bigvee_{i=1}^m F_i(x_1) = \{\{y_1, y_2\}, \{y_2\}, \{y_2, y_3\}\}$,
$\bigvee_{i=1}^m F_i(x_2) = \{\{y_2\}, \{y_3\}, \{y_3\}\}$,
$\bigvee_{i=1}^m F_i(x_3) = \{\{y_1, y_3\}, \{y_1\}, \{y_1, y_3\}\}$.

Furthermore, for any $y \in \bigcup_{i=1}^m F_i(x_2)$ means that $y \in \{y_2, y_3\} = \bigcup_{i=1}^m F_i(x_2)$ and $y \in \bigvee_{i=1}^m F_i(x_2)$ means that $y \in F_1(x_2)$ or $y \in F_2(x_2)$ or $y \in F_3(x_2)$.

Then, Based on this results, we discuss the relationship for rough approximation of a fuzzy set between $\bigcup_{i=1}^m F_i$, $\bigcap_{i=1}^m F_i$ and $\bigvee_{i=1}^m F_i$.

Definition 3.3 Let (U, V, R) be multigranulation approximation space over two universes. $\mathfrak{R}$ is a family binary compatibility relation between U and V induced by binary mapping family $F_i : U \to 2^V, u \mapsto \{v \in V|(u, v) \in R_i\}$, $R_i \in \mathfrak{R}, i = 1, 2, \cdots, m$. For any $A \in F(V)$, the lower approximation $\underline{R}_{\bigcup_{i=1}^m R_i}(A)$ and upper approximation $\overline{R}_{\bigcup_{i=1}^m R_i}(A)$ of A with respect to approximation space over two universes (U, V, R) are defined as follows, respectively.

$\underline{R}_{\bigcup_{i=1}^m R_i}(A)(x) = \min\{A(y)|y \in \bigcup_{i=1}^m F_i(x), y \in V\}$, $x \in U$;
$\overline{R}_{\bigcup_{i=1}^m R_i}(A)(x) = \max\{A(y)|y \in \bigcup_{i=1}^m F_i(x), y \in V\}$, $x \in U$.

By the definition of $\underline{R}_{\bigcup_{i=1}^m R_i}(A)$ and $\overline{R}_{\bigcup_{i=1}^m R_i}(A)$, the following assertions are clear.

Proposition 3.5 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$ and $x \in U$, there are

(1) $\underline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)(x) \geq \underline{R}_{\bigcup_{i=1}^m R_i}(A)(x)$,
(2) $\overline{\mathfrak{R}}^O_{\sum_{i=1}^m R_i}(A)(x) \leq \overline{R}_{\bigcup_{i=1}^m R_i}(A)(x)$,

Proof. It is easy to know that there is $y \in \vee_{i=1}^{m} F_i(x) \Rightarrow y \in \cup_{i=1}^{m} F_i(x)$ for any $y \in V$. Then the results in (1) and (2) are clear.

Proposition 3.6 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$ and $x \in U$, there are

(1) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) \leq \underline{R}_{\bigcap_{i=1}^{m} R_i}(A)(x)$,

(2) $\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) \geq \overline{R}_{\bigcap_{i=1}^{m} R_i}(A)(x)$,

Proof. It is easy to know that there is $y \in \cap_{i=1}^{m} F_i(x) \Rightarrow y \in \vee_{i=1}^{m} F_i(x)$ for any $y \in V$. Then the results in (1) and (2) are clear.

Moreover, it is easy to verify that the strict inequalities are established under the condition of $\bigcap_{i=1}^{m} F_i \subset \bigvee_{i=1}^{m} F_i$. This can be verified by the follow example.

Example 3.3 (Continued from Example 3.1) From example 3.1, we have the following results:

$\bigcap_{i=1}^{m} F_i(x_1) = \{y_1, y_2\} \cap \{y_2\} \cap \{y_2, y_3\}\{y_2\} = \{y_2\}$,

$\bigvee_{i=1}^{m} F_i(x_1) = \{y_1, y_2\} \vee \{y_2\} \vee \{y_2, y_3\}$.

That is,

$\bigvee_{i=1}^{m} F_i(x_1) = \{\{y_1, y_2\}, \{y_2\}, \{y_2, y_3\}\}$.

At the same time, it is to verify that $\bigcap_{i=1}^{m} F_i(x_1) \subset \bigvee_{i=1}^{m} F_i(x_1)$.

Furthermore, we have

$\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{3} R_i}(A)(x_1) = \min\{A(y_1), A(y_2), A(y_3)\} = 0.4$,

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{3} R_i}(A)(x_1) = \max\{A(y_1), A(y_2), A(y_3)\} = 0.9$,

and

$\underline{R}_{\bigcap_{i=1}^{m} R_i}(A)(x) = \min\{A(y_2)\} = 0.6$,

$\overline{R}_{\bigcap_{i=1}^{m} R_i}(A)(x) = \max\{A(y_2)\} = 0.6$.

So, there are

$\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{3} R_i}(A)(x_1) = 0.4 < \underline{R}_{\bigcap_{i=1}^{m} R_i}(A)(x) = \min\{A(y_2)\} = 0.6$,

and

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{3} R_i}(A)(x_1) = 0.9 > \overline{R}_{\bigcap_{i=1}^{m} R_i}(A)(x) = \max\{A(y_2)\} = 0.6$.

This completes the example.

Meanwhile, we can prove the following properties for the approximation operators of optimistic multigranulation rough fuzzy set over two universes.

Theorem 3.2 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A, B \in F(V)$. Then there are

(1) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) \subseteq \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) \subseteq U$,

(2) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A^c) = (\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))^c$,

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A^c) = (\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))^c$,

(3) If $A \subseteq B$, then $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m}}(A) \subseteq \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m}}(B)$,

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) \subseteq \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(B)$.

Theorem 3.3 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A_j \in F(V)(j = 1, 2, \cdots, k)$. Then there are

(1) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(\bigcap_{j=1}^{k} A_j) \subseteq \bigcap_{j=1}^{k} \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A_j)$,

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(\bigcup_{j=1}^{k} A_j) \supseteq \bigcup_{j=1}^{k} \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A_j)$,

(2) $\underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(\bigcup_{j=1}^{k} A_j) \supseteq \bigcup_{j=1}^{k} \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A_j)$,

$\overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(\bigcap_{j=1}^{k} A_j) \subseteq \bigcap_{j=1}^{m} \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A_j)$.

The proof of Theorem 3.2 and Theorem 3.3 can be directly derived by the definition.

3.2. *Pessimistic multigranulation rough fuzzy set over two universes*

Similar to the optimistic multigranulation rough fuzzy set, we also can systematically investigate the properties and the relationship between pessimistic and variable precision multigranulation rough fuzzy set over two universes and the existing rough set models.

Because there are the same way to discuss the properties and the similar results for pessimistic and variable precision multigranulation rough fuzzy set with optimistic multigranulation rough fuzzy set over two universes, then we only present the background of management science and the definitions for these two models.

We first establish pessimistic multigranulation rough fuzzy set model over two universes.

Pessimistic multigranulation rough fuzzy set model describes the decision making process of conservative type decision-makers or risk-averse decision-makers. For example, given an optimal alternative selecting decision making of multiple criteria decision making problem, a risk-averse decision-maker will select those alternatives that satisfy all considered criteria as the optimal selection decision making. Actually, many decision making problems in practice of management science such as medical diagnosis, pattern recognition and emergency decision making of unconventional emergency events require that the optimal decision making results must be satisfy all required conditions because the characteristic of the decision making problem itself. These types of decision making problems only can be depicted by using pessimistic multigranulation rough fuzzy set model when the decision making objects are a fuzzy concept. So, research on pessimistic multigranulation rough fuzzy set is necessity and valuable.

Definition 3.4 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$. The pessimistic lower approximation $\underline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ and pessimistic upper approximation $\overline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ of fuzzy set A in $(U, V, \mathfrak{R})$ are defined as follows, respectively.

$\underline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)(x) = \min\{A(y)|y \in \bigwedge_{i=1}^{m} F_i(x), y \in V\}, \quad x \in U;$

$\overline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)(x) = \max\{A(y)|y \in \bigwedge_{i=1}^{m} F_i(x), y \in V\}, \quad x \in U.$

The minimum and maximum become inf and sup when universe U and V are infinite set. Where $\bigwedge_{i=1}^{m} F_i(x) = F_1(x) \wedge F_2(x) \wedge \cdots \wedge F_m(x)$ and then $y \in F_1(x) \wedge F_2(x)$ means that $y \in F_1(x)$ and $y \in F_2(x)$.

It is easy to know that $\underline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ and $\overline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ are two fuzzy sets of universe U. Furthermore, A is called a definable fuzzy set on multigranulation approximation space over two universes $(U, V, \mathfrak{R})$ when $\underline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A) = \overline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$. Otherwise, we call the set-pair $(\underline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A))$ pessimistic multigranulation rough fuzzy set over two universes.

Meanwhile, from the point of view of risk decision making with uncertainty, the lower approximation $\underline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ can be regarded as the " min – min " rule and the upper approximation $\overline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ can be regarded as the " max – min " rule. So, the upper approximation of any fuzzy set with respect to multigranulation approximation space over two universes is the type of pessimistic decision model in traditional risk decision making with uncertainty.

3.3. Variable precision multigranulation rough fuzzy set over two universes

Let us re-consider the definition of optimistic multigranulation rough fuzzy set and pessimistic multigranulation rough fuzzy set over two universes. On the one hand, for any $x \in U$ and $y \in V$, if there exists at least one binary mapping $F_i(x)$ in m binary mappings $F_i(i = 1, 2, \cdots, m)$ which satisfies $y \in F_i(x)$, then we can obtain the lower and upper approximations of a given fuzzy set A of universe V with respect to multigranulation approximation space over two universes. This constructs the optimistic multigranulation rough fuzzy set model over two universes. On the other hand, for any $x \in U$ and $y \in V$, if and only if all of the m binary mappings $F_1(x), F_2(x), \cdots, F_m(x)$ satisfy $y \in F_i(x)(i = 1, 2, \cdots, m)$, then we can obtain the lower and upper approximations of a given fuzzy set A of universe V with respect to multigranulation approximation space over two universes. This constructs the pessimistic multigranulation rough fuzzy set model over two universes.

It can be easily seen that optimistic multigranulation rough fuzzy set and pessimistic multigranulation rough fuzzy set over two universes only consider two extremely cases of a decision making process: completely risk-preferring and completely risk-averse. Although both optimistic multigranulation rough fuzzy set and pessimistic multigranulation rough fuzzy set over two universes can deal with many uncertainty decision making problems in practice, there has a limitation for modelling the uncertainty decision making problems because there only two extremely cases are considered in the existing two models. So, an improved model of the established multigranulation rough fuzzy set over two universes is needed. In the following, we define a new version of the multigranulation rough fuzzy set over two universes by introducing the precision parameter in the existed models, i.e., variable precision multigranulation rough fuzzy set model over two universes.

We first give the characteristic function for binary mapping between two different universes.

Definition 3.5 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. F is any arbitrary binary mapping from universe U to V. For any $x \in U$ and $y \in V$, the characteristic function of F with respect to universe V is defined as follows:

$$\chi_V^F(y) = \begin{cases} 1, & y \in F(x); \\ 0, & \text{Others}; \end{cases}$$

By the definition of characteristic function for any binary mapping from universe U to V, we present variable precision multigranulation rough fuzzy set over two universes as follows.

Definition 3.6 Let $(U, V, \mathfrak{R})$ be multigranulation approximation space over two universes. For any $A \in F(V)$. The variable precision lower approximation $\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)$ and upper approximation $\overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)$ of fuzzy set A in $(U, V, \mathfrak{R})$ are defined as follows, respectively.

$$\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)(x) = \min\{A(y) | \tfrac{\sum_{i=1}^m \chi_V^{F_i}(y)}{m} \geq \alpha, y \in V\}, \quad x \in U;$$
$$\overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)(x) = \max\{A(y) | \tfrac{\sum_{i=1}^m \chi_V^{F_i}(y)}{m} \geq \alpha, y \in V\}, \quad x \in U.$$

The minimum and maximum become inf and sup when universe U and V are infinite set.

It is easy to know that $\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)$ and $\overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)$ are two fuzzy sets of universe U. Furthermore, A is called a definable fuzzy set on multigranulation approximation space over two universes $(U, V, \mathfrak{R})$ when $\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A) = \overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)$. Otherwise, we call $(\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A), \overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A))$ variable precision multigranulation rough fuzzy set over two universes.

At the same time, based on the definition of variable precision multigranulation rough fuzzy set over two universes, the following results are clear.

Remark 3.2 If $\alpha = \frac{1}{m}$, for any $x \in U$, then we have

$$\begin{aligned} &\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)(x) = \min\{A(y) | \tfrac{\sum_{i=1}^m \chi_V^{F_i}(y)}{m} \geq \tfrac{1}{m}, y \in V\} \\ &= \min\{A(y) | \textstyle\sum_{i=1}^m \chi_V^{F_i}(y) \geq 1, y \in V\} \\ &= \min\{A(y) | y \in \vee_{i=1}^m F_i(x), y \in V\} \\ &= \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x), \end{aligned}$$

$$\begin{aligned} &\overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)(x) = \max\{A(y) | \tfrac{\sum_{i=1}^m \chi_V^{F_i}(y)}{m} \geq \tfrac{1}{m}, y \in V\} \\ &= \max\{A(y) | \textstyle\sum_{i=1}^m \chi_V^{F_i}(y) \geq 1, y \in V\} \\ &= \max\{A(y) | y \in \vee_{i=1}^m F_i(x), y \in V\} \\ &= \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)(x). \end{aligned}$$

So, there is

$$\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m F_i}(A) = \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m F_i}(A) \;\; and \;\; \overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A) = \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A).$$

That is, variable precision multigranulation rough fuzzy set degenerates into optimistic multigranulation rough fuzzy set over two universes. Moreover, we also can verify that

$$\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m F_i}(A) = \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^m F_i}(A) \;\; and \;\; \overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A) = \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^m R_i}(A)$$

hold for $\alpha \in (0, \frac{1}{m}]$.

Remark 3.3 If $\alpha = 1$, for any $x \in U$, then we have

$$\begin{aligned} &\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)(x) = \min\{A(y) | \tfrac{\sum_{i=1}^m \chi_V^{F_i}(y)}{m} \geq 1, y \in V\} \\ &= \min\{A(y) | \textstyle\sum_{i=1}^m \chi_V^{F_i}(y) \geq m, y \in V\} \\ &= \min\{A(y) | y \in \wedge_{i=1}^m F_i(x), x \in U, y \in V\} \\ &= \underline{\mathfrak{R}}^{P}_{\sum_{i=1}^m R_i}(A)(x), \end{aligned}$$

$$\begin{aligned} &\overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A)(x) = \max\{A(y) | \tfrac{\sum_{i=1}^m \chi_V^{F_i}(y)}{m} \geq 1, y \in V\} \\ &= \max\{A(y) | \textstyle\sum_{i=1}^m \chi_V^{F_i}(y) \geq m, y \in V\} \\ &= \max\{A(y) | y \in \wedge_{i=1}^m F_i(x), y \in V\} \\ &= \overline{\mathfrak{R}}^{P}_{\sum_{i=1}^m R_i}(A)(x). \end{aligned}$$

So, there is

$$\underline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m F_i}(A) = \underline{\mathfrak{R}}^{P}_{\sum_{i=1}^m F_i}(A) \;\; and \;\; \overline{\mathfrak{R}}^{\alpha}_{\sum_{i=1}^m R_i}(A) = \overline{\mathfrak{R}}^{P}_{\sum_{i=1}^m R_i}(A).$$

That is, variable precision multigranulation rough fuzzy set degenerates into pessimistic multigranulation rough fuzzy set over two universes.

From the point of view of risk decision making with uncertainty, both the lower approximation $\underline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ and upper approximation $\overline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ can be regarded as the weighted decision rule (or the equality possibility rule) in the theory of traditional risk decision making with uncertainty, which the weight is $\frac{1}{m}$ for every criterion $F_i (i = 1, 2, \cdots, m)$ and parameter α is the expected precision for considered decision making problem. Meanwhile, it can be easily seen that variable precision multigranulation rough fuzzy set model has included optimistic and pessimistic multigranulation rough fuzzy set models over two universes. That is, optimistic and pessimistic multigranulation rough fuzzy set models are the special case of variable precision multigranulation rough fuzzy set model over two universes. Also, we can easy to know that variable precision multigranulation rough fuzzy set model describes the gradually changing process from optimistic multigranulation rough fuzzy set to pessimistic multigranulation rough fuzzy set over two universes when the values of parameter α increasing in interval $[\frac{1}{m}, 1]$. Therefore, the limitation which only considers the completely risk-preferring and completely risk-averse cases for a decision making problem in optimistic and pessimistic multigranulation rough fuzzy set over two universes is improved by introducing precision parameter α. Then variable precision multigranulation rough fuzzy set models over two universes can adapt to solve any kind of decision making problems with uncertainty in management science.

So far there are several approaches to rough set approximations in a multigranulation space and also many various generalized multigranulation rough set models are established [26, 28, 29, 42, 49, 61]. In the existing literatures, Yao and She [61] analysis the existing studies of the multigranulation rough set models and propose a unified framework to classify and compare the existing studies about various multigranulation rough set models. They classify two distinct directions/classes of research on rough set approximations in multigranulation spaces. One class constructs approximations based on the combined granulations. The other class combines approximations from individual granulations [61]. It is easy to know that the three multigranulation rough set models defined in this section belong to the latter class. The aims of this paper is try to construct a kind of multiple criteria group decision making model and method by using the idea of multigranulation rough set theory. In the defined three multigranulation models of this section, the individual granulation is regarded as the criterion of the group decision making problem. So, different decision-makers will select different criteria flexibility according to their preferences and domain knowledge in the process of group decision making. This also is the advantage of the approach to multiple criteria group decision making problems comparing to the traditional group decision making methods [2, 5, 16, 47]. Therefore, the three models given in this paper are defined by the way of constructing a family of approximations, i.e., the combination of approximation from individual granulations.

4. Multigranulation fuzzy rough set over two universes

This section considers rough approximation of a fuzzy concept with respect to multigranulation fuzzy approximation space. We will give three different multigranulation fuzzy rough set models over two universes based on the corresponding decision making background of management science, respectively.

In Section 3, we discuss rough approximation of a fuzzy concept in multigranulation approximation space over two universes which determined by a family of binary compatibility mappings between universe U and V. However, there may be a family of binary fuzzy relation over two different universes because the increasing complexity of the socio-economic environment in reality. Then, multigranulation fuzzy approximation space over two universes arisen. This section will focus on rough approximation of a fuzzy concept in multigranulation fuzzy approximation space over two universes, then optimistic multigranulation fuzzy rough set, pessimistic multigranulation fuzzy rough set and two types of variable precision multigranulation fuzzy rough sets on multigranulation fuzzy approximation space over two universes will be established in detail.

First, we present the definition of optimistic and pessimistic multigranulation fuzzy rough set models over two universes, respectively.

Definition 4.1 Let triple ordered set (U, V, R) be multigranulation fuzzy approximation space over two universes. For any $A \in F(V)$ and $x \in U$, the optimistic lower approximation $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ and optimistic upper approximation $\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ of A in (U, V, R) are defined as follows, respectively.

$$\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) = \bigvee_{i=1}^{m} \bigwedge_{y \in V} [(1 - R_i(x, y)) \vee A(y)],$$

$$\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) = \bigwedge_{i=1}^{m} \bigvee_{y \in V} [R_i(x, y) \wedge A(y)].$$

Particular, the operator $\bigvee$ and $\bigwedge$ become inf and sup when universe U and V are infinite set.

It is easy to know that $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ and $\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$ are two fuzzy sets of universe U. Furthermore, A is called a definable fuzzy set on multigranulation fuzzy approximation space over two universes (U, V, R) when $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) = \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)$. Otherwise, we call $(\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))$ optimistic multigranulation fuzzy rough set over two universes.

Definition 4.2 Let triple ordered set (U, V, R) be multigranulation fuzzy approximation space over two universes. For any $A \in F(V)$ and $x \in U$, the pessimistic lower approximation $\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ and pessimistic upper approximation $\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$ of A in (U, V, R) are defined as follows, respectively.

$$\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)(x) = \bigwedge_{i=1}^{m} \bigwedge_{y \in V} [(1 - R_i(x, y)) \vee A(y)],$$

$$\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)(x) = \bigvee_{i=1}^{m} \bigvee_{y \in V} [R_i(x, y) \wedge A(y)].$$

Similarly, we call A is a definable fuzzy set on multigranulation fuzzy approximation space over two universes (U, V, R) when $\underline{\mathrm{R}}_{\sum_{i=1}^{m} R_i}(A) = \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$, and we call $(\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A))$ pessimistic multigranulation fuzzy rough set over two universes when $\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A) \neq \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)$.

Remark 4.1 If R is a family of binary compatibility relation between U and V, $\forall x \in U$, then

$$\begin{aligned}
\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) &= \bigvee_{i=1}^{m} \bigwedge_{y\in V} [(1-R_i(x,y)) \vee A(y)] \\
&= \bigwedge_{y\in \bigvee_{i=1}^{m} F_i(x)} A(y) \\
&= \underline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x), \\
\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x) &= \bigwedge_{i=1}^{m} \bigvee_{y\in V} [R_i(x,y) \wedge A(y)] = \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A^c)(x)^c \\
&= \bigvee_{y\in \bigvee_{i=1}^{m} F_i(x)} A(y) \\
&= \overline{\mathfrak{R}}^{O}_{\sum_{i=1}^{m} R_i}(A)(x); \\
\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)(x) &= \bigwedge_{i=1}^{m} \bigwedge_{y\in V} [(1-R_i(x,y)) \vee A(y)] \\
&= \bigwedge_{y\in \bigwedge_{i=1}^{m} F_i(x)} A(y) \\
&= \underline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)(x), \\
\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)(x) &= \bigvee_{i=1}^{m} \bigvee_{y\in V} [R_i(x,y) \wedge A(y)] = \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A^c)(x)^c \\
&= \bigvee_{y\in \bigwedge_{i=1}^{m} F_i(x)} A(y) \\
&= \overline{\mathfrak{R}}^{P}_{\sum_{i=1}^{m} R_i}(A)(x).
\end{aligned}$$

That is, optimistic and pessimistic multigranulation fuzzy rough set models degenerate into optimistic and pessimistic multigranulation rough fuzzy set models over two universes, respectively. Therefore, we illustrate the relationship between multigranulation rough fuzzy set and multigranulation fuzzy rough set over two universes: the optimistic and pessimistic multigranulation rough fuzzy sets are the special case of the optimistic and pessimistic multigranulation fuzzy rough sets over two universes.

Note that if $R_1 = R_2 = \cdots = R_m$, then optimistic and pessimistic multigranulation fuzzy rough sets over two universes will degenerate into fuzzy rough set over two universes [21, 56]; Furthermore, optimistic and pessimistic multigranulation fuzzy rough sets over two universes will degenerate into the classical fuzzy rough set on single universe [7, 21, 61, 68] when there are $R_1 = R_2 = \cdots = R_m$ and $U = V$.

So, we establish the relationship between multigranulation fuzzy rough set over two universes and the existing rough set models, i.e., multigranulation fuzzy rough set is a natural extension of the existing rough set models. Moreover, the following theorems characterize the relationship between optimistic and pessimistic multigranulation fuzzy rough set and fuzzy rough set over two universes [53].

Theorem 4.1 Let triple ordered set (U, V, R) be multigranulation fuzzy approximation space over two universes. For any $A \in F(V)$, we have

(1) $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) = \bigcup_{i=1}^{m} \underline{R_i}(A)$, $\quad \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) = \bigcap_{i=1}^{m} \overline{R_i}(A)$.

(2) $\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A) = \bigcap_{i=1}^{m} \underline{R_i}(A)$; $\quad \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A) = \bigcup_{i=1}^{m} \overline{R_i}(A)$.

Proof. It can be easily derived from Definition 4.1, 4.2 and Definition 4 of Ref. [45].

By Definition 4.1 and 4.2, the following properties for the approximation operators of optimistic and pessimistic multigranulation fuzzy rough sets over two universes are clear.

Theorem 4.2 Let triple ordered set (U, V, R) be multigranulation fuzzy approximation space over two universes. For any $A, B \in F(V)$, and $A \subseteq B$, we have

(1) $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) \subseteq U$, $\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A), \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A) \subseteq U$,

(2) $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A^c) = (\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))^c$, $\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A^c) = (\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A))^c$,

(3) $\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A^c) = (\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A))^c$, $\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A^c) = (\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A))^c$,

(4) $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m}}(A) \subseteq \underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m}}(B)$, $\qquad \underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m}}(A) \subseteq \underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m}}(B)$,
$\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A) \subseteq \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(B)$, $\qquad \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A) \subseteq \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(B)$.

Theorem 4.3 Let (U, V, R) be multigranulation approximation space over two universes. For any $A_j \in F(V)(j = 1, 2, \cdots, k)$. Then there are

(1) $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(\bigcap_{j=1}^{k} A_j) \subseteq \bigcap_{j=1}^{k} \underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A_j)$,
$\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(\bigcap_{j=1}^{k} A_j) \subseteq \bigcap_{j=1}^{k} \underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A_j)$,

(2) $\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(\bigcup_{j=1}^{k} A_j) \supseteq \bigcup_{j=1}^{k} \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A_j)$,
$\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(\bigcup_{j=1}^{k} A_j) \supseteq \bigcup_{j=1}^{k} \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A_j)$,

(3) $\underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(\bigcup_{j=1}^{k} A_j) \supseteq \bigcup_{j=1}^{k} \underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A_j)$,
$\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(\bigcup_{j=1}^{k} A_j) \supseteq \bigcup_{j=1}^{k} \underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A_j)$,

(4) $\overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(\bigcap_{j=1}^{k} A_j) \subseteq \bigcap_{j=1}^{m} \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m} R_i}(A_j)$,
$\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(\bigcap_{j=1}^{k} A_j) \subseteq \bigcap_{j=1}^{m} \overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m} R_i}(A_j)$.

The proof of Theorem 4.2 and 4.3 can be directly derived by Definition 4.1 and 4.2.

Next, we present variable precision multigranulation fuzzy rough set model over two universes. The model will be constructed by introducing a precision parameter into the multigranulation fuzzy rough set model over two universes. Then we will present two types of variable precision multigranulation fuzzy rough set model by introducing precision parameter into optimistic and pessimistic multigranulation fuzzy rough set models, respectively.

Definition 4.3 Let triple ordered set (U, V, R) be multigranulation fuzzy approximation space over two universes. For any $A \in F(V)$ and parameter $\alpha \in [0, 1)$, the $\alpha-$lower approximation $(I)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m} R_i}(A)$ and $\alpha-$upper approximation $(I)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m} R_i}(A)$ of A in (U, V, R) with parameter α are defined as follows, respectively.

$$(I)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m} R_i}(A)(x) = \bigvee_{i=1}^{m}[\bigwedge_{A(y)\leq\alpha}((1-R_i(x,y))\vee\alpha)\wedge \bigwedge_{A(y)>\alpha}((1-R_i(x,y)) \vee A(y))], x \in U, y \in V,$$

$$(I)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m} R_i}(A)(x) = \bigwedge_{i=1}^{m}[\bigvee_{A(y)\geq 1-\alpha}(R_i(x,y)\wedge(1-\alpha))\vee \bigvee_{A(y)<1-\alpha}(R_i(x,y) \wedge A(y))], x \in U, y \in V.$$

The $\alpha-$lower approximation $(I)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m} R_i}(A)$ and $\alpha-$upper approximation $(I)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m} R_i}(A)$ are two fuzzy sets of universe U. We call A a definable fuzzy set on multigranulation fuzzy approximation space over two universes (U, V, R) with parameter α when $(I)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m} R_i}(A) = (I)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m} R_i}(A)$, and we call the set-pair

$((I)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A), (I)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A))$ the type-I variable precision multigranulation fuzzy rough set over two universes.

Definition 4.4 Let triple ordered set (U, V, R) be multigranulation fuzzy approximation space over two universes. For any $A \in F(V)$ and parameter $\alpha \in [0,1)$, the $\alpha-$lower approximation $(II)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)$ and $\alpha-$upper approximation $(II)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)$ of A in (U, V, R) with parameter α are defined as follows, respectively.

$$(II)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)(x) = \bigwedge_{i=1}^{m}[\bigwedge_{A(y)\le\alpha}((1-R_i(x,y))\vee\alpha)\wedge \bigwedge_{A(y)>\alpha}((1-R_i(x,y))\vee A(y))], x\in U, y\in V,$$

$$(II)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)(x) = \bigvee_{i=1}^{m}[\bigvee_{A(y)\ge 1-\alpha}(R_i(x,y)\wedge(1-\alpha))\vee \bigvee_{A(y)<1-\alpha}(R_i(x,y)\wedge A(y))], x\in U, y\in V.$$

The $\alpha-$lower approximation $(II)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)$ and $\alpha-$upper approximation $(II)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)$ are two fuzzy sets of universe U. We call A a definable fuzzy set on multigranulation fuzzy approximation space over two universes (U, V, R) with parameter α when $(II)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A) = (II)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)$, and we call the set-pair $((II)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A), (II)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A))$ the type-II variable precision multigranulation fuzzy rough set over two universes.

Remark 4.2 If $\alpha = 0$, for any $x \in U, y \in V$, then we there are

$$\begin{aligned}
(I)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)(x) &= \bigvee_{i=1}^{m}[\bigwedge_{A(y)\le\alpha}((1-R_i(x,y))\vee\alpha)\wedge \bigwedge_{A(y)>\alpha}((1-R_i(x,y))\vee A(y))]\\
&= \bigvee_{i=1}^{m}[\bigwedge_{y\in V}((1-R_i(x,y))\vee 0)\wedge \bigwedge_{A(y)>0}((1-R_i(x,y))\vee A(y))]\\
&= \bigvee_{i=1}^{m}[\bigwedge_{y\in V}(1-R_i(x,y))\wedge \bigwedge_{y\in V}((1-R_i(x,y))\vee A(y))]\\
&= \bigvee_{i=1}^{m}\bigwedge_{y\in V}((1-R_i(x,y))\vee A(y))\\
&= \underline{\mathrm{R}}^{O}_{\sum_{i=1}^{m}R_i}(A)(x),\\
(I)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)(x) &= \bigwedge_{i=1}^{m}[\bigvee_{A(y)\ge 1-\alpha}(R_i(x,y)\wedge(1-\alpha))\vee \bigvee_{A(y)<1-\alpha}(R_i(x,y)\wedge A(y))]\\
&= \bigwedge_{i=1}^{m}[\bigvee_{A(y)\ge(1-0)}(R_i(x,y)\wedge(1-0))\vee \bigvee_{A(y)<(1-0)}(R_i(x,y)\wedge A(y))]\\
&= \bigwedge_{i=1}^{m}[\bigvee_{A(y)\ge 1}R_i(x,y)\vee \bigvee_{y\in V}(R_i(x,y)\wedge A(y))]\\
&= \bigwedge_{i=1}^{m}\bigvee_{y\in V}(R_i(x,y)\wedge A(y))]\\
&= \overline{\mathrm{R}}^{O}_{\sum_{i=1}^{m}R_i}(A)(x).
\end{aligned}$$

Similarly, we can obtain that

$$(II)\underline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)(x) = (II)\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m}R_i}(A)(x)$$

and

$$(II)\overline{\mathrm{R}}^{\alpha}_{\sum_{i=1}^{m}R_i}(A)(x) = (II)\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m}R_i}(A)(x)$$

when the precision parameter $\alpha = 0$.

That is, the type-I and type-II variable precision multigranulation fuzzy rough set models will degenerate into optimistic and pessimistic multigranulation fuzzy rough set models over two universes, respectively.

Remark 4.3 In general, the lower approximation included into the upper approximation for the above multigranulation rough set models if and only if binary fuzzy relation $R_i(i = 1, 2, 3)$ are serial relation over U and V.

In order to verify the validity of lower and upper approximations of optimistic multigranulation fuzzy rough set over two universes and illustrate the assertion of Remark 4.3, considering a numerical example as follows.

Example 4.1 Let $U = \{x_1, x_2, x_3\}$ and $V = \{y_1, y_2, y_3\}$ be two non-empty universes. $R_i(i = 1, 2, 3) \in F(U \times V)$ are three binary fuzzy relations between universe U and V, respectively.

$$R_1 = \begin{pmatrix} 0.2 & 0.5 & 0.2\\ 0.7 & 0.3 & 0.1\\ 0.4 & 0.3 & 0.6 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 0.3 & 0.6 & 0.2\\ 0.4 & 0.4 & 0.3\\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0.5 & 0.4 & 0.3\\ 0.8 & 0.5 & 0.3\\ 0.3 & 0.5 & 0.4 \end{pmatrix}$$

Suppose a fuzzy set on universe V as follows:

$$A = \frac{0.4}{y_1} + \frac{0.6}{y_2} + \frac{0.9}{y_3}.$$

Here we only present the results of pessimistic multigranulation fuzzy rough set and Type-I variable precision multigranulation fuzzy rough set over two universes by using this numerical example. Then we have

$$\begin{aligned}
\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m}R_i}(A)(x_1) &= ([(1-0.2)\vee 0.4]\wedge[(1-0.5)\vee 0.6]\\
&\wedge[(1-0.2)\vee 0.9])\wedge([(1-0.3)\vee 0.4]\\
&\wedge[(1-0.6)\vee 0.6]\wedge[(1-0.2)\vee 0.9])\\
&\wedge([(1-0.5)\vee 0.4]\wedge[(1-0.4)\vee 0.6]\\
&\wedge[(1-0.3)\vee 0.9])\\
&= 0.5,\\
\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m}R_i}(A)(x_1) &= [(0.2\wedge 0.4)\vee(0.5\wedge 0.6)\vee(0.2\wedge 0.9)]\\
&\vee[(0.3\wedge 0.4)\vee(0.6\wedge 0.6)\vee(0.2\wedge 0.9)]\\
&\vee[0.5\wedge 0.4)\vee(0.4\wedge 0.6)\vee(0.3\wedge 0.9)]\\
&= 0.5\vee 0.6\vee 0.4\\
&= 0.6.
\end{aligned}$$

Similarly, we can obtain the lower and upper approximations of A are

$$\underline{\mathrm{R}}^{P}_{\sum_{i=1}^{m}R_i}(A) = \frac{0.5}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3}$$

and

$$\overline{\mathrm{R}}^{P}_{\sum_{i=1}^{m}R_i}(A) = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3}.$$

Let $\alpha = 0.7$. We have

$(I)\underline{R}^{0.7}_{\sum_{i=1}^m R_i}(A)(x_1) = ([(1-0.2)\vee 0.7]\wedge[(1-0.5)\vee 0.7]$
$\wedge[(1-0.2)\vee 0.9])\wedge([(1-0.3)\vee 0.7]$
$\wedge[(1-0.6)\vee 0.7]\wedge[(1-0.2)\vee 0.9])$
$\wedge([(1-0.5)\vee 0.7]\wedge[(1-0.4)\vee 0.7]$
$\wedge[(1-0.3)\vee 0.9])$
$= 0.7,$

$(I)\overline{R}^{0.7}_{\sum_{i=1}^m R_i}(A)(x_1) = [(0.2\wedge(1-0.7))\vee(0.5\wedge(1-0.7)))$
$\vee(0.2\wedge(1-0.7))]\vee[(0.3\wedge(1-0.7))$
$\vee(0.6\wedge(1-0.7))\vee(0.2\wedge(1-0.7))]$
$\vee[0.5\wedge(1-0.7))\vee(0.4\wedge(1-0.7))$
$\vee(0.3\wedge(1-0.7))]$
$= 0.3.$

Then we have

$$(I)\underline{R}^{0.7}_{\sum_{i=1}^m R_i}(A) = \frac{0.7}{x_1}+\frac{0.7}{x_2}+\frac{0.7}{x_3}$$

and

$$(I)\overline{R}^{0.7}_{\sum_{i=1}^m R_i}(A) = \frac{0.3}{x_1}+\frac{0.3}{x_2}+\frac{0.3}{x_3}.$$

Meanwhile, it is easy to see that there are

$$\underline{R}^{P}_{\sum_{i=1}^m R_i}(A) \nsubseteq \overline{R}^{P}_{\sum_{i=1}^m R_i}(A),\ \ (I)\underline{R}^{0.7}_{\sum_{i=1}^m R_i}(A) \nsubseteq (I)\overline{R}^{0.7}_{\sum_{i=1}^m R_i}(A)$$

because binary fuzzy relation $R_i (i = 1, 2, 3)$ are not serial relation over U and V.

Meanwhile, we also can prove the following properties hold for type-I and type-II variable precision multigranulation fuzzy rough sets over two universes.

Theorem 4.4 Let triple ordered set (U, V, R) be multigranulation fuzzy approximation space over two universes. For any $A, B \in F(V)$ and $\alpha \in [0, 1), A \subseteq B$, we have

(1) $(I)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A), (I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A) \subseteq U$,
$(II)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A), (II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A) \subseteq U$,

(2) $(I)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A^c) = ((I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A))^c$,
$(II)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A^c) = ((II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A))^c$,

(3) $(I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A^c) = ((I)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A))^c$,
$(II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A^c) = ((II)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A))^c$,

(4) $(I)\underline{R}^{\alpha}_{\sum_{i=1}^m}(A) \subseteq (I)\underline{R}^{\alpha}_{\sum_{i=1}^m}(B)$,
$(II)\underline{R}^{\alpha}_{\sum_{i=1}^m}(A) \subseteq (II)\underline{R}^{\alpha}_{\sum_{i=1}^m}(B)$,
$(I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A) \subseteq (I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(B)$,
$(II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A) \subseteq (II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(B)$.

Theorem 4.5 Let (U, V, R) be multigranulation approximation space over two universes. For any $A_j \in F(V) (j = 1, 2, \cdots, k)$. Then there are

(1) $(I)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(\bigcap_{j=1}^k A_j) \subseteq \bigcap_{j=1}^k (I)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A_j)$,
$(II)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(\bigcap_{j=1}^k A_j) \subseteq \bigcap_{j=1}^k (II)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A_j)$,

(2) $(I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(\bigcup_{j=1}^k A_j) \supseteq \bigcup_{j=1}^k (I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A_j)$,
$(II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(\bigcup_{j=1}^k A_j) \supseteq \bigcup_{j=1}^k (II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A_j)$,

(3) $(I)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(\bigcup_{j=1}^k A_j) \supseteq \bigcup_{j=1}^k (I)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A_j)$,
$(II)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(\bigcup_{j=1}^k A_j) \supseteq \bigcup_{j=1}^k (II)\underline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A_j)$,

(4) $(I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(\bigcap_{j=1}^k A_j) \subseteq \bigcap_{j=1}^m (I)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A_j)$,
$(II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(\bigcap_{j=1}^k A_j) \subseteq \bigcap_{j=1}^m (II)\overline{R}^{\alpha}_{\sum_{i=1}^m R_i}(A_j)$.

The proof of Theorem 4.4 and 4.5 can be directly derived by Definition 4.3 and 4.4.

So far we have defined the models of multigranulation rough fuzzy set and multigranulation fuzzy rough set systematically by approximating of a fuzzy concept on multigranulation approximation space and multigranulation fuzzy approximation space over two universes, respectively. By the above discussion, the family of arbitrary compatibility binary relation is the special case of arbitrary fuzzy binary relation between universe U and V, then multigranulation fuzzy rough set has included multigranulation rough fuzzy set over two universes. Therefore, we establish the theory of multigranulation fuzzy rough set over two universes.

As far as the optimistic and pessimistic decision models defined in this section, Li and Zhou [19] also proposed the idea of optimistic and pessimistic decisions based on decision-theoretic rough set [60]. From the viewpoint of decision making with uncertainty, the idea of optimistic and pessimistic decision for a given decision-maker is similar for both Li and Zhou's model and the models in this paper. The differences are Li and Zhou's decision model based on Bayesian risk decision and three-way decision with single granularity but the models in this paper lay on multiple granularity and the traditional two-way decision.

5. Multigranulation fuzzy rough set over two universes based multiple criteria group decision making method

The increasing complexity of the socio-economic environment, operational research, and industrial engineering force humans to tackle problems crossing many disciplines. Group decision making (GDM)[3, 5], as one of effectively approaches to handle the complexity decision making problems, is defined as a decision problem in which several experts provide their judgment over a set of alternatives. The aim is to reconcile (or comprehensive) differences of opinion expressed by individual experts to find an alternative (or set of alternatives) that is most acceptable by the group of experts as a whole. In a complex society, group decision making (GDM) processes must inevitably take many criteria (or factors) into account. Thus, research on group decision making (GDM) that explicitly incorporates multiple criteria has been a major direction, and has made significant progress with the rapid development of operations research, management science, systems engineering, and other disciplines. Hwang and Lin [16] first study to explore systematically how multiple criteria could be used in group decision making (GDM).

In general, multiple criteria group decision making problems (MCGDM) involve selecting or ranking from all of the feasible alternatives among multiple, conflicting, and interactive criteria. For example, in a decision recruitment problem for engaging a new young employee, the alternatives are the candidates and the criteria are some characteristics useful to give a comprehensive evaluation of the candidates such as educational degree, professional experience, age and job interview. At the same time, several experts are invited to give the comprehensive evaluation for all candidates according to the given criteria in advance. Then, aggregating all comprehensive evaluation given by the experts based on a determined method and then obtain the ranking for all candidates. Finally, the expected candidate with the highest ranking will be selected, i.e., the optimal decision making is given for this multiple criteria group decision making problem.

In this section, we try to establish a new approach to multiple criteria group decision making problems based on multigranulation fuzzy rough set over two universes. We present the basic description of a multiple criteria group decision making problem under the framework of multigranulation over two universes, and then give a general decision making methodology for multiple criteria group decision making problem by using the multigranulation fuzzy rough set theory over two universes.

5.1. Problem statement

We firstly give the basic description of considered multiple criteria group decision making problem in this paper. We present the description by using a multiple criteria group decision making problem in the case of clothes ranking.

Let $V = \{x_1, x_2, \cdots, x_m\}$ be the criteria set which $x_i (i = 1, 2, \cdots, m)$ are m given criteria, e.g., x_1 denotes pattern and color, x_2 denotes style, x_3 denotes durability, $\cdots$, and x_m denotes price. Let $U = \{y_1, y_2, \cdots, y_n\}$ be the decision set (i.e., the evaluation for the criteria), in which y_1 denotes very welcome, y_2 denotes welcome, y_3 denotes less welcome, $\cdots$, and y_n denotes not welcome. Suppose that $R_1, R_2, \cdots, R_k$ are k invited experts in group. Like the classical group decision making, every expert provides his evaluation for all criteria $x_i (x_i \in V)$ with respect to decision set elements $y_j (y_j \in U)$. Generally speaking, the evaluation $R_1, R_2, \cdots, R_k$ are fuzzy relation between criteria set V and decision evaluation set U, i.e., there are $R_1, R_2, \cdots, R_k \in F(U \times V)$. That is, $R_l(y_j, x_i)(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n; l = 1, 2, \cdots, k)$ is the evaluation of criteria x_i with evaluation element y_j given by expert l according to their experience and professional knowledge of himself. Let A be a category customer with right weight for each criterion in V. Obviously, A is a fuzzy set of criteria set V. Then the decision making for this multiple criteria group decision making problem is how to obtain the evaluation of this particular costume for this category customer.

In the following, we give an approach to decision making for this kind of multiple criteria group decision problem with the above described characteristic by using the theory of multigranulation fuzzy rough set over two universes. In this paper, we use the model of optimistic multigranulation fuzzy rough set over two universes to present the decision method for multiple criteria group decision making. Actually, all models of optimistic multigranulation fuzzy rough set, pessimistic multigranulation fuzzy rough set and variable precision multigranulation fuzzy rough set over two universes established in Section 4 can be used to discuss the above multiple criteria group decision making problem.

5.2. Decision making methodology

Firstly, we construct the multigranulation fuzzy decision information systems over two universes for the considered multiple criteria group decision problem.

From the description of the multiple criteria group decision making problem in Section 5.1, we know that the judgment of every invited expert provides a binary fuzzy relation between criteria set and decision set. Then there is a family of binary fuzzy relation R between criteria set V and decision set U given by all experts, i.e., $R_l \in \mathrm{R}, l = 1, 2, \cdots, k$. So, we obtain multigranulation fuzzy decision information systems over two universes (U, V, R) for the multiple criteria group decision making problem.

Secondly, we calculate optimistic multigranulation fuzzy lower approximation $\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$ and optimistic multigranulation fuzzy upper approximation $\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$ for the given category customer A (described by a fuzzy set of universe V) with respect to multigranulation fuzzy decision information systems over two universes (U, V, R).

As is discussed in Section 4, both $\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$ and $\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$ are fuzzy set of decision set U. Then we can obtain the ranking of the given category customers with respect to decision set, i.e., $\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)(y_j)$ and $\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)(y_j)$, $j = 1, 2, \cdots, n$. Denote

$$\sum_{l=1}^{k} R_l(A) = \lambda \underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A) + (1-\lambda)\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A) \qquad \lambda \in [0, 1].$$

It is easy to know that $\sum_{l=1}^{k} R_l(A) \in F(U)$, i.e., $\sum_{l=1}^{k} R_l(A)$ is a fuzzy set on decision set U.

Finally, based on the value of $\sum_{l=1}^{k} R_l(A)$, we give the ranking for the given category customer by using the principle of maximum membership in Zadeh's fuzzy set theory.

Remark 5.1 It can be easily obtained

$$\sum_{l=1}^{k} R_l(A) = \underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$$

when $\lambda = 1$ and

$$\sum_{l=1}^{k} R_l(A) = \overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$$

when $\lambda = 0$. Then we can present an interpretation of the decision rule given in above according to the definition of $\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$ and $\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$ as follows.

From the point of view of risk decision making with uncertainty,

$$\sum_{l=1}^{k} R_l(A) = \underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$$

can be regarded as the " min – min " rule,

$$\sum_{l=1}^{k} R_l(A) = \overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$$

can be regarded as the " max – min " rule and

$$\sum_{l=1}^{k} R_l(A) = \lambda \underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A) + (1-\lambda)\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A) \quad (0 < \lambda < 1)$$

can be regarded the compromise rule with a right weight λ.

In practice, the parameter λ also reflects the preference of decision-maker for risk of decision making. Generally speaking, the larger the value of parameter λ when decision-maker is risk-preferring. The smaller the value of parameter λ when decision-maker is risk-averse. So, the value of parameter λ is given by decision-maker's preference or the empirical studies in advance.

Therefore, we establish an approach to multiple criteria group decision making by using the theory of multigranulation fuzzy rough set over two universes. The application of this method will be given by using a clothes ranking decision making problem.

5.3. Algorithm for the proposed multiple criteria group decision making method

In this section, we present the algorithm for the established method of considered multiple criteria group decision making problem in Section 5.1.

Input Multigranulation fuzzy decision information systems over two universes (U, V, R).

Output The ranking of the given category customer.

Step 1 Computing multigranulation lower approximation $\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$ and multigranulation upper approximation $\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$ of fuzzy subset A of universe V about multigranulation fuzzy decision information systems over two universes (U, V, R);

Step 2 Determining the value of λ;

Step 3 Computing $\sum_{l=1}^{k} R_l(A)$;

Step 4 Present the ranking according to the decision principle given in Section 5.2.

5.4. A test example

In this section, we consider a multiple criteria group decision making problem in the case of clothes to illustrate the decision method proposed in Section 5.2.

Let $V = \{x_1, x_2, x_3, x_4\}$ be criteria set, in which x_1 denotes pattern and color, x_2 denotes style, x_3 denotes durability and x_4 denotes price. Let $U = \{y_1, y_2, y_3, y_4\}$ be decision set (i.e., the evaluation for the criteria), in which y_1 denotes very welcome, y_2 denotes welcome, y_3 denotes less welcome and y_4 denotes not welcome.

Suppose that R_1, R_2 and R_3 are three invited experts. They provides his evaluation for all criteria $x_i (x_i \in V)(i = 1, 2, 3, 4)$ with respect to decision set elements $y_j (y_j \in U)(j = 1, 2, 3, 4)$. As is discussed in Section 4.1, the evaluation R_1, R_2 and R_3 are fuzzy relation between criteria set V and decision evaluation set U. i.e., there are $R_1, R_2, R_3 \in F(V \times U)$.

So, we construct multigranulation fuzzy decision information systems over two universes (U, V, R) for this multiple criteria group decision making problem.

Suppose three experts present their judgment (the binary fuzzy matrix R_1, R_2 and R_3) for the criteria and decision set are as follows:

$$R_1 = \begin{pmatrix} 0.1 & 0.4 & 0.2 & 0.5 \\ 0.3 & 0.2 & 0.6 & 0.1 \\ 0.4 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.3 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 0.3 & 0.4 & 0.1 & 0.5 \\ 0.1 & 0.2 & 0.4 & 0.1 \\ 0.3 & 0.1 & 0.1 & 0.3 \\ 0.3 & 0.3 & 0.4 & 0.1 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0.2 & 0.5 & 0.4 & 0.3 \\ 0.5 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.3 \end{pmatrix}$$

Let A and B be two different category customers with right weights for each criterion in V are as follows:

$$A = \frac{0.4}{x_1} + \frac{0.35}{x_2} + \frac{0.15}{x_3} + \frac{0.1}{x_4}$$

and

$$B = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.4}{x_4}.$$

Taking $\lambda = 0.2$. Then, the multiple criteria group decision making of the first category customer A can be obtained as follows:

$$\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A) = \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{0.6}{y_3} + \frac{0.7}{y_4}$$

and

$$\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A) = \frac{0.3}{y_1} + \frac{0.4}{y_2} + \frac{0.35}{y_3} + \frac{0.3}{y_4}.$$

So, we have

$$\sum_{l=1}^{k} R_l(A) = 0.2\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A) + (1 - 0.2)\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(A)$$

$$= \frac{0.38}{y_1} + \frac{0.44}{y_2} + \frac{0.4}{y_3} + \frac{0.38}{y_4}.$$

Then, according to the principle of maximum membership in Zadeh's fuzzy set theory, this particular costume is "welcome" for the first category customer.

Similarly, we can obtain the following results for the second category customer B.

$$\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(B) = \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{0.6}{y_3} + \frac{0.7}{y_4}$$

and

$$\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(B) = \frac{0.2}{y_1} + \frac{0.2}{y_2} + \frac{0.2}{y_3} + \frac{0.3}{y_4}.$$

So, we have

$$\sum_{l=1}^{k} R_l(B) = 0.2\underline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(B) + (1 - 0.2)\overline{\mathrm{R}}^{O}_{\sum_{l=1}^{k} R_l}(B)$$

$$= \frac{0.3}{y_1} + \frac{0.28}{y_2} + \frac{0.28}{y_3} + \frac{0.38}{y_4}.$$

Then, according to the principle of maximum membership in Zadeh's fuzzy set theory, this particular costume is "not welcome" for the second category customer.

If necessary, we can normalize $\sum_{l=1}^{k} R_l(A)$ and $\sum_{l=1}^{k} R_l(B)$ as follows, respectively.

$$\sum_{l=1}^{k} R_l(A) = \frac{0.2375}{y_1} + \frac{0.275}{y_2} + \frac{0.25}{y_3} + \frac{0.2375}{y_4},$$

$$\sum_{l=1}^{k} R_l(B) = \frac{0.242}{y_1} + \frac{0.226}{y_2} + \frac{0.226}{y_3} + \frac{0.306}{y_4}.$$

As is well known, the consensus measurement and aggregation of all preference relations given by whole experts [2, 44, 45, 47, 65] are two important issues of multiple criteria group decision making problem. So far many different methods for handling consensus measurement and preference relation aggregation for multiple criteria group decision making problems have established in the past few years [2, 15, 43]. This paper makes a tentative investigation to deal with multiple criteria group decision making problems by using the multigranulation fuzzy rough set theory over two universes. The important contribution is establishing a new way to handle with expert preference (or expert opinions) by using the multiple granularity approach to group decision making problems.

6. Conclusions

Rough set theory over two universes and multigranulation rough set are two interesting generalizations of Pawlak rough set theory. This paper studies multigranulation fuzzy rough set theory over two universes by combing this two generalized theories. Comparing the existing literatures [14, 18, 20 − 26, 36 − 40, 49, 55, 56, 66], the new contributions can be concluded as follows: (1) We systematically discuss rough approximation of a fuzzy concept on multigranulation approximation space and multigranulation fuzzy approximation space under the framework of two universes. At the same time, we present two types of multigranulation fuzzy rough set models over two universes. The existing literatures [38 − 40, 45, 56] are discuss the approximations of a crisp concept about the multigranulation approximation space based on the single universe, or discuss the single-granulation rough set models over two universes [42, 44 − 46]. As is discussed in Section 3.1, the multigranulation fuzzy rough set over two universes are more generalized model which have included the existing models on the single universe and the single-granulation models over two universes. (2) Different reasonable background description from the point of view of the classical risk decision making with uncertainty and the detailed theory basis investigation are given for the proposed multigranulation models over two universes. (3) A new approach to multiple criteria group decision making problem based on multigranulation fuzzy rough set over two universes is established. The published papers are mainly focus on the discussion of the mathematical properties and the feature selection for the multigranulation models [14, 18, 20 − 26, 42, 49, 55, 56, 66] and the uncertainty decision making based on rough set models over two universes [42 − 46], less effort on the discussion of multiple criteria group decision making methods based on multigranulation rough sets on single universe or two different universes. This paper makes a meaningful attempt to apply the multigranulation fuzzy rough set over two universes to multiple criteria group decision making problems with uncertainty. As is well known, the opinions or preferences of all decision-makers (or experts) aggregation is the critical step for the traditional group decision making methods [2, 5, 16, 47]. In our proposed decision making method, every decision-maker is regarded as a binary granulation defined on two different universes and then all opinions given by decision-makers are aggregated by using the multigranulation lower and upper approximations under the framework of two universes, and then a compromise optimal proposal is obtained based on the multigranulation lower and upper approximations over two universes. So, the multigranulation fuzzy rough set approach to multiple criteria group decision making under the framework of two different universes provides another way to aggregate the preferences of decision-makers. Therefore, the proposed decision making method also presents a new tool and perspective to explore group decision making problems in reality.

For further study, the primary theory and characterization of multigranulation fuzzy rough sets over two universes are needed. On the other hand, the attribute reduction of multigranulation fuzzy approximation space over two universes should be studied and detailed experimental investigation and comparison with existing approaches should be discussed. Although this paper focuses on the basic theory of multiple criteria group decision making principal with multigranulation fuzzy rough set over two universes, it is recommended that the further improved of the proposed method to apply more complexity decision making problems as mentioned in above possible areas and the real-life data be used to test the approach established in this paper.

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References

[1] G. Abu-Donia, Multi knowledge based rough approximations and applications, Knowledge-Based Syst. 26(1)(2012) 20-29.

[2] A. Altuzarra, J. Moreno-Jiménez, M. Salvador, Consensus building in AHP-group decision making: A Bayesian approach, Oper. Res. 58(6)(2010)1755-1773.

[3] K. Arrow. Social choice and individual values, 2nd edn.

Wiley, New York, 1963.

[4] M. Aquil Khan, M. Banerjee, Formal reasoning with rough sets in multiple-source approximation systems, Int. J. Approx. Reason. 49(2008)466-477.

[5] Y. Chen, D. Kilgour, K. Hipel, A decision rule aggregation approach to multiple criteria-multiple participant sorting, Group Decis. Negot. 21(2012)727-745.

[6] G. Carey, L. Rob, Incorporating the rough sets theory into travel demand analysis, Tourism Management. 24(2003) 511-517.

[7] A. Dimitras, R. Slowinski, R. Susmaga, C. Zopounidis, Business failure prediction using rough sets, Euro. J. Oper. Res. 114(1999)263-280.

[8] H. Dou, X. Yang, X. Song, H. Yu, W. Wu, J Yang, Decision-theoretic rough set: a multicost strategy, Knowledge-Based Syst. 91(2016)71-83.

[9] Z. Fu, X. Wu, C. Guan, X. Sun, K. Ren, Towards efficient multi-keyword fuzzy search over encrypted outsourced data with accuracy improvement, IEEE Trans. Inf. Foren. Sec. 11(12)(2016)2706-2716.

[10] Z.J. Fu, K. Ren, J.G. Shu, X.M. Sun, F.X. Huang, Enabling personalized search over encrypted outsourced data with efficiency improvement, IEEE Trans. Parall. Distr. 27(9)(2016) 2546-2559.

[11] S. Greco, B. Matarazzo, R. Slowinski, Rough sets theory for multicriteria decision analysis, Euro. J. Oper. Res. 129(2001)11-47.

[12] B. Gu, V.S. Sheng, A robust regularization path algorithm for ν-support vector classification, IEEE Trans. Neur. Net. Learn. DOI: 10.1109/TNNLS.2016.2527796, 2016

[13] B. Gu, V.S. Sheng, Z. Wang, Derek Ho, S. Osman, S. Li, Incremental learning for ν-Support Vector Regression, Neur. Net. 67(2015)140-150.

[14] B. Huang, C. Guo, Y. Zhang, H. Li, X. Zhou, Intuitionistic fuzzy multigranulation rough sets, Inf. Sci. 277(2014)299-320.

[15] D. Hong, C. Choi, Multicriteria fuzzy decision-making problems based on vague set theory, Fuzzy Sets Syst. 114(2000) 103-113.

[16] C. Hwang, M. Lin, Group decision making under multiple criteria, Lecture Notes in Economics Math. Syst. *Springer, Berlin,* 1987.

[17] R. Jensen, Q. Shen, Fuzzy-rough sets assisted attribute selection, IEEE Trans. Fuzzy Syst. 15(1)(2007)73-89.

[18] S. Kumar, H. Inbarani, Optimistic multi-granulation rough set based classification for medical diagnosis, Procedia Comput. Sci. 47(2015)374-382.

[19] H. Li, X. Zhou, Risk decision making based on decision-theoretic rough set: A three-way view decision model, Int. J. Comput. Intell. Syst. 4(1)(2011)1-11.

[20] J. Liang, F. Wang, C. Dang, Y. Qian, An efficient rough feature selsction algorithm with a multi-granulation view, Int. J. Approx. Reason. 53(2012)912-926.

[21] G. Lin, Y. Qian, J. Liang, NMGRS: neighborhood-based multigranulation rough sets, Int. J. Approx. Reason. 53(7)(2012)1080-1093.

[22] C. Liu, D. Miao, J. Qian On multi-granulation covering rough sets, Int. J. Approx. Reason. 55(6)(2014)1404-1418.

[23] C. Liu, W. Wang, Covering fuzzy rough set based on multi-granulations. *Int. Conf. on Uncertainty Reason. and Knowl. Eng.* pp. 146-149, 2012.

[24] S. Liu, F. Chan, S. Chung, A study of distribution center location based on the rough sets and interactive multi-objective fuzzy decision theory, Robotics and Computer Integrated Manufacturing, 27(2011)426-433.

[25] G. Lin, J. Liang, Y. Qian, An information fusion approach by combining multigranulation rough sets and evidence theory, Inf. Sci. 314(2015)184-199.

[26] Y. Lin, J. Li, P. Lin, G. Lin, J. Chen, Feature selection via neighborhood multi-granulation fusion, Knowledge-Based Syst. 67(2014)162-168.

[27] X.Li, H.Yi, S. Liu, Rough sets and matroids from a lattice-theoretic viewpoint, Inf. Sci. 342(2016)37-52.

[28] G. Lin, J. Liang, Y. Qian, J. Li, A fuzzy multigranulation decision-theoretic approach to multi-source fuzzy information systems, Knowledge-Based Syst. 91(2016)102-113.

[29] J. Li, Y. Ren, C. Mei, Y. Qian, X. Yang. A comparative study of multigranulation rough sets and concept lattices via rule acquisition, Knowledge-Based Syst. 91(C)(2015)152-164.

[30] J. Li, X.L. Li, B. Yang, X.M. Sun, Segmentation-based image copy-move forgery detection scheme, IEEE Trans. Foren. Sec. 10(3)(2015)507-518.

[31] N. Morsi, M. Yakout, Axiomatics for fuzzy rough sets, Fuzzy Sets Syst. 100(1-3)(1998)327-342.

[32] J. Moreno-Jim*é*nez, J. Aguar*ó*n, M. Escobar, A. Tur*ó*n, The multicriteria procedural rationality on Sideman, Euro. J. Oper. Res. 119(2)(1999)388-403.

[33] J.M. Ma, Y. Yao, Rough set approximations in multi-granulation fuzzy approximation spaces, Fundamenta Informaticae 142(1-4)(2015)145-160.

[34] T. Ma, J. Zhou, M. Tang, Y. Tian, A. Al-Dhelaan, M. Al-Rodhaan, S. Lee, Social network and tag sources based augmenting collaborative recommender system, IEeE Trans. Inf. Systs. 98(4)(2015)902-910.

[35] Z. Pawlak, Rough sets, Int. J. Inf. Sci. 11(5)(1982)341-356.

[36] D. Pei, Z. Xu, Rough set models on two universes, Int. J. General Syst. 33(5)(2004)569-581.

[37] Y. Qian, J. Liang, C. Dang, Incomplete multigranulation rough set, IEEE Trans. Syst. Man Cyber. Part A: Systems and Humans 20(2010)420-431.

[38] Y. Qian, S. Li, J. Liang, Z. Shi, F. Wang, Pessimistic rough set based decisions: a multigranulation fusion strategy, Inf. Sci. 264(2014)196-210.

[39] Y. Qian, J, Liang, Y. Yao, C. Dang, MGRS: a multi-granulation rough set, Inf. Sci. 180(2010)949-970.

[40] Y. Qian, H. Zhang, Y. Sang, J. Liang, Multigranulation decision-theoretic rough sets, Int. J. Approx. Reason. 55(1)(2014)225-237.

[41] R. Slowinski, D. Vanderpooten, A generalized definition of rough approximations based on similarity, IEEE Trans. Knowl. Data Eng. 12(2)(1990)331-336.

[42] B. Sun, W. Ma, Multigranulation rough set theory over two universes, J. Intell. Fuzzy Syst. 28(3)(2015)1251-1269.

[43] B. Sun, W. Ma, Fuzzy rough set over multi-universes and its application in decision making, J. Intell. Fuzzy Syst. DOI(2016) 10.3233/JIFS-151977.

[44] B. Sun, W. Ma, H. Zhao, Rough set-based conflict analysis model and method over two universes, Inf. Sci. 372(2016)111-125.

[45] B. Sun, W. Ma, Rough approximation of a preference relation by multi-decision dominance for a multi-agent conflict analysis problem, Inf. Sci. 315(2015)39-53.

[46] B. Sun, W. Ma, X.T. Chen, Fuzzy rough set on probabilistic approximation space over two universes and its application to emergency decision-making, Expert Syst. 32(4)(2015) 507-521.

[47] B. Sun, W. Ma, An approach to consensus measurement of linguistic preference relations in multi-attribute group decision making and application, Omega 51(2015)83-92.

[48] P. Sawicki, Z. Jacek, Technical diagnostic of a fleet of vehicles using rough set theory, Euro. J. Oper. Res. 193(2009) 891-903.

[49] Y. She, X. He, On the structure of the multigranulation rough set model, Knowledge-Based Syst. 36(2012)81-92.

[50] E. Straszecka, Combining uncertainty and imprecision in models of medical diagnosis, Inf. Sci. 176(20)(2007)3026-3059.

[51] E. Tay Francis, L. Shen, Economic and financial prediction using rough sets model, Euro. J. Oper. Res. 141(2002) 641-659.

[52] S. Wong, L. Wang, Y. Yao, On modeling uncertainty with interval structures, Comput. Intell. 11(1993)406-426.

[53] W. Wu, W. Zhang, Constructive and axiomatic approaches of fuzzy approximation operators, Inf. Sci. 159(2004)233-254.

[54] W. Wu, Y. Leung, Theory and applications of granular labeled partitions in multi-scale decision tables, Inf. Sci. 181(18)(2011)3878-3897.

[55] W. Xu, W. Sun, X. Zhang, W. Zhang, Multiple granulation rough set approach to ordered information systems, Int. J. General Syst. 41(2012)475-501.

[56] W. Xu, Q. Wang, S. Luo, Multi-granulation fuzzy rough sets, J. Int. Fuzzy Syst. 26(3)(2014)1323-1340.

[57] Z.H. Xia, X.H. Wang, X.M. Sun, Q.S. Liu, N.X. Xiong, Steganalysis of LSB matching using differences between nonadjacent pixels, Multimed. Tools Appl. 75(4)(2016)1947-1962.

[58] Y. Yao, S. Wong, L. Wang, A non-numeric approach to uncertain reasoning, Int. J. General Syst. 23(1995)343-359.

[59] Y. Yao, Probabilistic rough set approximations, Int. J. Approx. Reason. 49(2)(2008)255-271.

[60] Y. Yao, Three-way decisions with probabilistic rough sets, Inf. Sci. 180(3)(2010)341-353.

[61] Y. Yao, Y. She, Rough set models in multigranulation spaces, Inf. Sci. 327(2015)40-56.

[62] Y. Yao, B. Zhou, Two Bayesian approaches to rough sets, Eur. J. Oper. Res. 251(2016)904-917.

[63] Y. Yao, Three-way decisions and cognitive computing, Cogn. Comput. 8(4)(2016)543-554.

[64] Y. Yao, The two sides of the theory of rough sets, Knowledge-Based Syst. 80(2015)67-77.

[65] R. Yager, On the fusion of imprecise uncertainty measures using belief structures, Inf. Sci. 181(2011)3199-3209.

[66] X. Yang, X. Song, Z. Chen, J. Yang, On multigranulation rough sets in incomplete information system, Int. J. Mach. Learn. Cyber. 3(2012)223-232.

[67] C.S. Yuan, X.M. Sun, R. Lv, Fingerprint liveness detection based on multi-scale LPQ and PCA, China Commun. 13(7)(2016)60-65.

[68] J. Zhang, T. Li, H. Chen, Composite rough sets for dynamic data mining, Inf. Sci. 257(2014)81-100.

[69] X. Zhang, D. Miao, C. Liu, M. Le, Constructive methods of rough approximation operators and multigranulation rough sets, Knowledge-Based Syst. 91(2016)114-125.

[70] J. Zhan, Q. Liu., B.J. Dawaz. A new rough set theory: rough soft hemirings, J. Intell. Fuzzy Syst. 28(4)(2015)1687-1697.

[71] Y. Zheng, B. Jeon, D. Xu, Q.M. Jonathan Wu, H. Zhang, Image segmentation by generalized hierarchical fuzzy C-means algorithm, J. Int. Fuzzy Syst. 28(2)(2015)961-973.

[72] Z. Zhou, Y. Wang, Q.M. Jonathan Wu, C.N. Yang, X.M. Sun. Effective and efficient global context verification for image copy detection, IEEE Trans. Inf. Foren. Sec. 12(1)(2017) 48-63.