

On Characterizing Hierarchies of Granulation Structures via Distances

Xibei Yang^{*†}

School of Computer Science and Engineering, Jiangsu University of Science and Technology

Zhenjiang, 212003, P.R. China

yangxibei@hotmail.com

Yuhua Qian[‡]

Key Laboratory of Comp. Intel. and Chinese, Information Processing of Ministry of Education

Taiyuan, 030006, P.R. China

jinchengqyh@126.com

Jingyu Yang

School of Computer Science and Technology, Nanjing University of Science and Technology

Nanjing, 210093, P.R. China

yangjy@mail.njust.edu.cn

Abstract. Hierarchy plays a crucial role in the development of the granular computing. In this paper, three different hierarchies are considered for judging whether a granulation structure is finer or coarser than another one. The first hierarchy is based on the set containment of information granulations, the second hierarchy is based on the cardinal numbers of information granulations while the third hierarchy is based on the sum of cardinal numbers of information granulations. Through introducing set distance and knowledge distance, we investigate the algebraic lattices, in which the derived partial orders are corresponding to the three different hierarchies, respectively. From the viewpoint of distance, these results look forward to provide a more comprehensible perspective for the study of hierarchies on granulation structures.

Keywords: granular computing, granulation structure, hierarchy, knowledge distance, lattice, set distance

*Address for correspondence: No. 2, Mengxi Road, School of Computer Science and Engineering, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu Province, 212003, P.R. China

†Also works: School of Comp. Sci. and Technology, Nanjing University of Sci. and Technology, Nanjing, 210093, P.R. China

‡Also works: School of Computer and Information Technology, Shanxi University, Taiyuan, 030006, P.R. China

1. Introduction

Granular computing[2, 3, 5, 6, 19, 21, 22, 23, 25, 26], was firstly proposed by Zadeh. It is not only a new concept but also an emerging computing paradigm of information processing. Though there is not a formal definition for granular computing yet, the aim of granular computing is to study the construction, representation, and interpretation of information granules, as well as utilization of information granules for problem solving, human thinking and information processing. Presently, the theories of fuzzy set, rough set[9, 10, 11, 12] and quotient space[27] are considered as three major mathematical tools for granular computing, these theories have also been demonstrated to be useful in knowledge discovery, pattern recognition, machine learning, decision support, medical diagnosis and so on.

It is well-known that hierarchy plays a crucial role in the development of granular computing. Such hierarchy can be used to define the finer or coarser relationships between different granulation structures. From the view point of set-theoretic approach[1, 24], a granulation structure can be referred to as the collection of neighborhoods (information granules) in terms of a binary relation in this paper. Given a hierarchy, different levels of granulation structures represent different granulated views of the universe, different granulated views can be linked together through such hierarchy, all levels are (partially) ordered according to their granularity[24]. For example, maps can be hierarchically organized into different scales, from large to small and vice versa[18]. A bigger scale represents a coarser granulated view of the map, while a smaller scale represents a finer granulated view of the map. As a matter of fact, Yao[20] said that there are at least two ways for the construction of a hierarchy, namely the top-down and the bottom-up approaches. Keet[8] showed that characterizing levels of granularity must adhere to the same type of granularity as their perspective and that each level is in exactly one perspective. Moreover, hierarchy can be used to research information granulation, measure knowledge content, measure the significance of an attribute and their applications in an information system[4, 16, 17]. For instance, without loss of generality, information measures and information granulation measures are monotonic with the hierarchical varieties of granulation structures.

Presently, with respect to different requirements, several types of (partial) orders[13, 15] have been proposed to investigate hierarchies on granulation structures. In this paper, we are mainly focusing on three typical hierarchies. The first one is based on the set containment, the second one is based on the cardinal number of neighborhood while the third one is based on the sum of cardinal numbers of neighborhoods. It can be observed obviously that the first hierarchy is a special case of the second hierarchy and the second hierarchy is a special case of the third hierarchy.

The purpose of this paper is to study the hierarchies on granulation structures through distances, which include set distance[4] and knowledge distance[14], respectively. Set distance is used to characterize the difference between two finite classical sets while the knowledge distance is used to characterize the difference between two granulation structures. Such two concepts have been successfully used to characterize rough set model, decision evaluation, rough measure, information measure and information granulation in Ref.[4]. Moreover, knowledge distance can also be used to solve the problem that the same uncertainty measurements (e.g. information granulations and knowledge entropies) are obtained by different granulation structures. For such reason, distance provides us a powerful tool for dealing with uncertainties.

To facilitate our discussion, we first present the basic notions of the granulation structure, set distance and knowledge distance in Section 2. In Section 3, the set with single element is considered as the frame of reference and then we define some operations on set distances to construct algebraic lattices, from

which the derived partial orders are corresponding to the hierarchies we focused on. In Section 4, the finest granulation structure is considered as the frame of reference and then we define some operations on knowledge distances to construct knowledge distances lattices, from which the derived partial orders are also corresponding to the hierarchies we focused on. Results are summarized in Section 5.

2. Preliminaries

In this section, we will review some basic concepts related to granular computing. Throughout this paper, we suppose that the universe U is a finite nonempty set.

2.1. Knowledge base and granulation structure

1. Let $U \neq \emptyset$ be a universe of discourse, \mathbf{R} is a family of equivalence relations on U , then the pair (U, \mathbf{R}) is referred to as a knowledge base[9];
2. If $P \subseteq \mathbf{R}$ and $P \neq \emptyset$, then $\bigcap P$ (intersection of all equivalence relations in P) is also an equivalence relation, and will be denoted by $IND(P)$, it is referred to as an indiscernibility relation over P in Pawlak's rough set theory.

$\forall R \in \mathbf{R}$, we use U/R to represent the family of the equivalence classes, which are derived from the equivalence relation R . Such equivalence classes are also referred to as categories of R . Therefore, $\forall x \in U$, $[x]_R$ is used to denotes a category (equivalence class) in terms of R , which contains x .

Suppose that $P \subseteq \mathbf{R}$, then $U/IND(P)$ is the family of the equivalence classes, which are generated from the equivalence relation $IND(P)$, each element in $U/IND(P)$ is referred to as a P -basic knowledge, $[x]_{IND(P)} = \{y \in U : (x, y) \in IND(P)\}$ is the equivalence class of $IND(P)$, which contains x . Through using a given knowledge base, Pawlak constructed a rough set of any subset on the universe as following definition shows.

Definition 2.1. [9] Let (U, \mathbf{R}) be a given knowledge base, in which $P \subseteq \mathbf{R}$, $\forall X \subseteq U$, the lower and upper approximations of X are:

$$\underline{IND(P)}(X) = \cup\{Y \in U/IND(P) : Y \subseteq X\} = \{x \in U : [x]_{IND(P)} \subseteq X\}; \quad (1)$$

$$\overline{IND(P)}(X) = \cup\{Y \in U/IND(P) : Y \cap X \neq \emptyset\} = \{x \in U : [x]_{IND(P)} \cap X \neq \emptyset\}. \quad (2)$$

Though Pawlak's rough set was proposed through an equivalence relation, it can also be generalized through some more general binary relations. In the following, we will assume that reflexive is necessary.

Given a universe U , suppose that \mathbf{B} is the set of all binary reflexive relations over U , if $A \subseteq \mathbf{B}$, then we denote the granulation structure by $K(A)$, which is derived from the intersection of binary relations in A , i.e. $\bigcap_{a \in A} a$.

To simplify our discussions, since $\bigcap_{a \in A} a$ is also a reflexive relation, then $\forall x \in U$, we assume that x has a neighborhood such that

$$[x]_A = \{y \in U : (x, y) \in a, \forall a \in A\}.$$

Formally, the granulation structure is defined as $K(A) = \{[x]_A : x \in U\}$. In equivalence relation case, each neighborhood $[x]_A$ (equivalence class) may be viewed as an information granule consisting of indistinguishable elements in the universe of discourse, the derived granulation structure is then a partition on the universe.

If \mathbf{B} is the set of all the binary reflexive relations over U , we denote $K(U)$ the set of all granulation structures on U in terms of \mathbf{B} , i.e.

$$K(U) = \{K(A) : A \subseteq \mathbf{B}\}.$$

In $K(U)$, there are two special cases: one is the discrete granulation structure, it is denoted by $\omega = \{\{x\} : x \in U\}$; the other is the indiscrete granulation structure, it is denoted by $\sigma = \{U : x \in U\}$ [14]. In discrete classification, any two objects in U can be distinguished from each other, from which we can conclude that we hold the maximal knowledge; while in indiscrete classification, any two objects in U cannot be distinguished from each other, from which we can conclude that we hold the minimal knowledge.

Definition 2.2. Let U be the universe of discourse, $\forall K(A), K(B) \in K(U)$, $K(A)$ is referred to as finer than $K(B)$ or $K(B)$ is referred to as coarser than $K(A)$ (denoted by $K(A) \preceq_1 K(B)$ or $K(B) \succeq_1 K(A)$) if and only if $[x]_A \subseteq [x]_B$ for each $x \in U$.

Obviously, in Definition 2.2, \preceq_1 is a partial order since it is reflexive, antisymmetric and transitive. Therefore, $(K(U), \preceq_1)$ is a partially ordered set.

Proposition 2.3. Let U be the universe of discourse, $\forall K(A) \in K(U)$, we have

$$\omega \preceq_1 K(A) \preceq_1 \sigma.$$

Proof:

It can be derived directly since $[x]_A$ is a reflexive neighborhood, which contains x . □

Definition 2.4. [13] Let U be the universe of discourse, $\forall K(A), K(B) \in K(U)$, $K(A)$ is referred to as finer than $K(B)$ or $K(B)$ is referred to as coarser than $K(A)$ (denoted by $K(A) \preceq_2 K(B)$ or $K(B) \succeq_2 K(A)$) if and only if $|[x]_A| \leq |[x]_B|$ for each $x \in U$ where $|X|$ is the cardinal number of set X .

Through Definition 2.4, we can see that \preceq_2 is not a partial order in the frame of $K(U)$ since it is not necessarily antisymmetric. For example, take for instance the following two granulation structures:

$$\begin{aligned} K(A) &= \{\{x_1, x_2\}, \{x_3, x_4\}\}; \\ K(B) &= \{\{x_1, x_3\}, \{x_2, x_4\}\}. \end{aligned}$$

By Definition 2.4, we have $K(A) \preceq_2 K(B)$ and $K(B) \preceq_2 K(A)$ while $K(A) \neq K(B)$. Obviously, \preceq_2 is reflexive and transitive, from which we know that \preceq_2 is a quasi-order.

Furthermore, let us consider the following two granulation structures:

$$\begin{aligned} K(A) &= \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}; \\ K(B) &= \{\{x_1, x_3\}, \{x_2, x_4, x_5\}, \{x_6\}\}. \end{aligned}$$

By Definition 2.4, we can see that $K(A) \not\preceq_2 K(B)$ since $|[x_6]_A| > |[x_6]_B|$; similarly, $K(B) \not\preceq_2 K(A)$ also holds since $|[x_2]_B| > |[x_2]_A|$. That is to say, the second hierarchy cannot be used to judge whether $K(A)$ is finer or coarser than $K(B)$. To solve such problem, a new order is defined in Definition 2.5.

Definition 2.5. Let U be the universe of discourse, $\forall K(A), K(B) \in K(U)$, $K(A)$ is referred to as finer than $K(B)$ or $K(B)$ is referred to as coarser than $K(A)$ (denoted by $K(A) \preceq_3 K(B)$ or $K(B) \succeq_3 K(A)$) if and only if $\sum_{x \in U} |[x]_A| \leq \sum_{x \in U} |[x]_B|$.

Through Definition 2.5, we also know that \preceq_3 is not a partial order in the frame of $K(U)$ since it is not necessarily antisymmetric. For example, suppose that $K(A) = \{\{x_1, x_2\}, \{x_3, x_4\}\}$, $K(B) = \{\{x_1, x_3\}, \{x_2, x_4\}\}$, by Definition 2.5, we have $K(A) \preceq_3 K(B)$ and $K(B) \preceq_3 K(A)$ while $K(A) \neq K(B)$. Nevertheless, \preceq_3 is reflexive and transitive, it is then a quasi-order.

Moreover, if $K(A) = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$, $K(B) = \{\{x_1, x_3\}, \{x_2, x_4, x_5\}, \{x_6\}\}$, then we have $\sum_{x \in U} |[x]_A| = 12 < 14 = \sum_{x \in U} |[x]_B|$, from which we can conclude that $K(A) \preceq_3 K(B)$, i.e. \preceq_3 can be used to compare two different granulation structures when both \preceq_1 and \preceq_2 do not work.

From discussions above, we know that \preceq_1 is a partial order while \preceq_2 and \preceq_3 are quasi-orders, it is not difficult to observe that the following implication hold for the above three orders:

$$\preceq_1 \implies \preceq_2 \implies \preceq_3 .$$

Since \preceq_1 is a partial order, we then can discuss the lattice structure of $K(U)$ in terms of \preceq_1 . However, since \preceq_2 and \preceq_3 are not necessarily partial orders, new approach has become a necessity to discuss the algebraic structures of $K(U)$ in terms of \preceq_2 and \preceq_3 . What the approach we will used in this paper is distance. In our approach, two different distances, set distance and knowledge distance are used, respectively. In the following, let us review such two distances briefly.

2.2. Set distance

The concept of set closeness between two classical sets is used to measure the degree of the sameness between sets. Let X and Y be two finite sets, the measure is defined by $H(X, Y) = \frac{|X \cap Y|}{|X \cup Y|}$ ($X \cup Y \neq \emptyset$). Obviously, the formula $1 - H(X, Y) = 1 - \frac{|X \cap Y|}{|X \cup Y|}$ can characterize the difference between two finite classical sets. It should be noticed that $1 - H(X, Y)$ can be interpreted through using the well-known Marczewski–Steinhaus metric[7], or MZ metric for short.

Definition 2.6. [4] Let U be the universe of discourse, $\forall X, Y \subseteq U$, the distance between X and Y is defined as:

$$d(X, Y) = 1 - \frac{|X \cap Y|}{|X \cup Y|}, \quad (3)$$

where $X \cup Y \neq \emptyset$.

Theorem 2.7. Let U be the universe of discourse, $\forall X, Y, Z \subseteq U$, we have the following properties about set distance:

1. Positive: $d(X, Y) \geq 0$ and $d(X, Y) = 0 \iff X = Y$;
2. Symmetric: $d(X, Y) = d(Y, X)$;
3. Triangle inequality:
 - (a) $d(X, Y) + d(X, Z) \geq d(Y, Z)$;
 - (b) $d(X, Y) + d(Y, Z) \geq d(X, Z)$;
 - (c) $d(X, Z) + d(Y, Z) \geq d(X, Y)$.

Through Theorem 2.7, we can see that (U, d) is a distance space.

2.3. Knowledge distance

Knowledge distance is a mathematical tool, which can be used to measure the difference between two different granulation structures. Such a concept was firstly proposed by Qian et al.[14] to solve the problem that the same uncertainty measurements (e.g. information granulations and knowledge entropies) are obtained by different granulation structures. The formal definition of knowledge distance is shown in Definition 2.8.

Definition 2.8. [14] Let U be the universe of discourse, $\forall K(A), K(B) \in K(U)$, the knowledge distance between $K(A)$ and $K(B)$ is denoted by $D(K(A), K(B))$ such that

$$D(K(A), K(B)) = \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus [x]_B|}{|U|}, \quad (4)$$

where $[x]_A \oplus [x]_B$ is the symmetric difference between $[x]_A$ and $[x]_B$, i.e. $[x]_A \oplus [x]_B = ([x]_A - [x]_B) \cup ([x]_B - [x]_A)$.

In Definition 2.8, $0 \leq D(K(A), K(B)) \leq 1 - \frac{1}{|U|}$ holds obviously.

Theorem 2.9. Let U be the universe of discourse, $\forall K(A), K(B), K(C) \in K(U)$, we have the following properties about knowledge distance:

1. Positive: $D(K(A), K(B)) \geq 0$ and $D(K(A), K(B)) = 0 \iff K(A) = K(B)$;
2. Symmetric: $D(K(A), K(B)) = D(K(B), K(A))$;
3. Triangle inequality:
 - (a) $D(K(A), K(B)) + D(K(A), K(C)) \geq D(K(B), K(C))$;
 - (b) $D(K(A), K(B)) + D(K(B), K(C)) \geq D(K(A), K(C))$;
 - (c) $D(K(A), K(C)) + D(K(B), K(C)) \geq D(K(A), K(B))$.

Through Theorem 2.9, we can see that $(K(U), D)$ is a distance space.

3. Algebraic lattices of set distances

In this section, we will investigate the hierarchies on granulation structures from the viewpoint of set distance. The main idea of our approach is to construct several algebraic lattices through set distance. By these algebraic lattices, the derived partial orders can be used to judge whether a granulation structure is finer or coarser than another one or not. To achieve such goal, we need a frame of reference. In this paper, the set with single element, i.e. $\{x\}$ is used.

Given a granulation structure $K(A) \in K(U)$, $\forall x \in U$, the set distance between neighborhood of x and $\{x\}$ is $d([x]_A, \{x\}) = 1 - \frac{|[x]_A \cap \{x\}|}{|[x]_A \cup \{x\}|}$. Since $[x]_A$ is at least a reflexive neighborhood, we then obtain $d([x]_A, \{x\}) = 1 - \frac{1}{|[x]_A|}$. We denote $d(K(A), \{x\})$ the set of all the set distances between neighborhoods in granulation structure $K(A)$ and $\{x\}$, i.e.

$$d(K(A), \{x\}) = \{d([x]_A, \{x\}) : x \in U\}.$$

For example, suppose that $K(A) = \{\{x_1, x_2\}, \{x_3, x_4\}\}$, then we obtain the set of the set distances such that $d(K(A), \{x\}) = \{0.5, 0.5, 0.5, 0.5\}$.

Since for each granulation structure in $K(U)$, we can obtain a set of the set distances, then all the sets of set distances is denoted by $d(K(U), \{x\})$ such that

$$d(K(U), \{x\}) = \{d(K(A), \{x\}) : K(A) \in K(U)\}.$$

Definition 3.1. Let U be the universe of discourse, $\forall K(A), K(B) \in K(U)$, we may define the following operations:

$$K(A) \sqcap_1 K(B) = K(A \cap B); \quad (5)$$

$$K(A) \sqcup_1 K(B) = K(A \cup B); \quad (6)$$

$$d(K(A), \{x\}) \sqcap_2 d(K(B), \{x\}) = \{\min\{d([x]_A, \{x\}), d([x]_B, \{x\})\} : x \in U\}; \quad (7)$$

$$d(K(A), \{x\}) \sqcup_2 d(K(B), \{x\}) = \{\max\{d([x]_A, \{x\}), d([x]_B, \{x\})\} : x \in U\}; \quad (8)$$

$$\sum_{x \in U} d(K(A)\{x\}) \sqcap_3 \sum_{x \in U} d(K(B), \{x\}) = \min \left\{ \sum_{x \in U} d(K(A)\{x\}), \sum_{x \in U} d(K(B), \{x\}) \right\}; \quad (9)$$

$$\sum_{x \in U} d(K(A)\{x\}) \sqcup_3 \sum_{x \in U} d(K(B), \{x\}) = \max \left\{ \sum_{x \in U} d(K(A)\{x\}), \sum_{x \in U} d(K(B), \{x\}) \right\}. \quad (10)$$

In Definition 3.1, operations \sqcap_1 and \sqcup_1 are defined on the knowledge structures; operations \sqcap_2 and \sqcup_2 are defined on the set distances between neighborhoods in granulation structure and single element x ; operations \sqcap_3 and \sqcup_3 are defined on the sums of set distances between neighborhoods in granulation structure and single element x .

Theorem 3.2. Let U be the universe of discourse,

1. $(K(U), \sqcap_1, \sqcup_1)$ is a lattice;
2. $(d(K(U), \{x\}), \sqcap_2, \sqcup_2)$ is a lattice;

3. $\left(\sum_{x \in U} d(K(U), \{x\}), \sqcap_3, \sqcup_3\right)$ is a lattice where

$$\sum_{x \in U} d(K(U), \{x\}) = \left\{ \sum_{x \in U} d(K(A), \{x\}) : K(A) \in K(U) \right\}.$$

Proof:

The proof of 1 has been shown in Ref. [14]. In the following, we only prove 2, the proof of 3 is similar to the proof of 2.

1. idempotent: $\forall d(K(A), \{x\}) \in d(K(U), \{x\})$, by Definition 3.1, we have

$$\begin{aligned} d(K(A), \{x\}) \sqcap_2 d(K(A), \{x\}) &= \{ \min\{d([x]_A, \{x\}), d([x]_A, \{x\})\} : x \in U \} \\ &= \{ d([x]_A, \{x\}) : x \in U \} \\ &= d(K(A), \{x\}). \end{aligned}$$

Similarity, it is not difficult to prove that $d(K(A), \{x\}) \sqcup_2 d(K(A), \{x\}) = d(K(A), \{x\})$.

2. commutative: $\forall d(K(A), \{x\}), d(K(B), \{x\}) \in d(K(U), \{x\})$, by Definition 3.1, we have

$$\begin{aligned} d(K(A), \{x\}) \sqcap_2 d(K(B), \{x\}) &= \{ \min\{d([x]_A, \{x\}), d([x]_B, \{x\})\} : x \in U \} \\ &= \{ \min\{d([x]_B, \{x\}), d([x]_A, \{x\})\} : x \in U \} \\ &= d(K(B), \{x\}) \sqcap_2 d(K(A), \{x\}). \end{aligned}$$

Similarity, it is not difficult to prove that $d(K(A), \{x\}) \sqcup_2 d(K(B), \{x\}) = d(K(B), \{x\}) \sqcup_2 d(K(A), \{x\})$.

3. associative: $\forall d(K(A), \{x\}), d(K(B), \{x\}), d(K(C), \{x\}) \in d(K(U), \{x\})$, by Definition 3.1, we have

$$\begin{aligned} &d(K(A), \{x\}) \sqcap_2 (d(K(B), \{x\}) \sqcap_2 d(K(C), \{x\})) \\ &= d(K(A), \{x\}) \sqcap_2 \{ \min\{d([x]_B, \{x\}), d([x]_C, \{x\})\} : x \in U \} \\ &= \{ \min\{d([x]_A, \{x\}), \min\{d([x]_B, \{x\}), d([x]_C, \{x\})\}\} : x \in U \} \\ &= \{ \min\{d([x]_A, \{x\}), d([x]_B, \{x\}), d([x]_C, \{x\})\} : x \in U \} \\ &= \{ \min\{\min\{d([x]_A, \{x\}), d([x]_B, \{x\})\}, d([x]_C, \{x\})\} : x \in U \} \\ &= \{ \min\{d([x]_A, \{x\}), d([x]_B, \{x\})\} : x \in U \} \sqcap_2 d(K(C), \{x\}) \\ &= (d(K(A), \{x\}) \sqcap_2 d(K(B), \{x\})) \sqcap_2 d(K(C), \{x\}). \end{aligned}$$

Similarity, it is not difficult to prove that $d(K(A), \{x\}) \sqcup_2 (d(K(B), \{x\}) \sqcup_2 d(K(C), \{x\})) = (d(K(A), \{x\}) \sqcup_2 d(K(B), \{x\})) \sqcup_2 d(K(C), \{x\})$.

4. absorption: $\forall d(K(A), \{x\}), d(K(B), \{x\}) \in d(K(U), \{x\})$, by Definition 3.1, we have

$$\begin{aligned} & d(K(A), \{x\}) \sqcap_2 (d(K(A), \{x\}) \sqcup_2 d(K(B), \{x\})) \\ &= d(K(A), \{x\}) \sqcap_2 \{max\{d([x]_A, \{x\}), d([x]_B, \{x\})\} : x \in U\} \\ &= \{min\{d([x]_A, \{x\}), max\{d([x]_A, \{x\}), d([x]_B, \{x\})\}\} : x \in U\} \\ &= \{d([x]_A, \{x\}) : x \in U\} \\ &= d(K(A), \{x\}). \end{aligned}$$

Similarity, it is not difficult to prove that $d(K(A), \{x\}) \sqcup_2 (d(K(A), \{x\}) \sqcap_2 d(K(B), \{x\})) = d(K(A), \{x\})$. □

Theorem 3.3. Let U be the universe of discourse,

1. $(K(U), \sqcap_1, \sqcup_1)$ is a distributive lattice;
2. $(d(K(U), \{x\}), \sqcap_2, \sqcup_2)$ is a distributive lattice;
3. $(\sum_{x \in U} d(K(U), \{x\}), \sqcap_3, \sqcup_3)$ is a distributive lattice .

Proof:

The proof of 1 has been shown in Ref. [14]. In the following, we only prove 2, the proof of 3 is similar to the proof of 2.

$\forall d(K(A), \{x\}), d(K(B), \{x\}), d(K(C), \{x\}) \in d(K(U), \{x\})$, by Definition 3.1, we have

$$\begin{aligned} & d(K(A), \{x\}) \sqcap_1 (d(K(B), \{x\}) \sqcup_1 d(K(C), \{x\})) \\ &= d(K(A), \{x\}) \sqcap_1 \{max\{d([x]_B, \{x\}), d([x]_C, \{x\})\} : x \in U\} \\ &= \{min\{d([x]_A, \{x\}), max\{d([x]_B, \{x\}), d([x]_C, \{x\})\}\} : x \in U\} \\ &= \{max\{min\{d([x]_A, \{x\}), d([x]_B, \{x\})\}, min\{d([x]_A, \{x\}), d([x]_C, \{x\})\}\} : x \in U\} \\ &= (d(K(A), \{x\}) \sqcap_2 d(K(B), \{x\})) \sqcup_2 (d(K(A), \{x\}) \sqcap_2 d(K(C), \{x\})). \end{aligned}$$

Similarity, it is not difficult to prove that $d(K(A), \{x\}) \sqcup_2 (d(K(B), \{x\}) \sqcap_2 d(K(C), \{x\})) = (d(K(A), \{x\}) \sqcup_2 d(K(B), \{x\})) \sqcap_2 (d(K(A), \{x\}) \sqcup_2 d(K(C), \{x\}))$. □

It is well-known that a partial order can be derived from an algebraic lattice. Through the three algebraic lattices we constructed in Theorem 3.2, three partial orders can be obtained such that:

1. $K(A) \sqsubseteq_1 K(B) \iff K(A) \sqcap_1 K(B) = K(A)$ or $K(A) \sqcup_1 K(B) = K(B)$;
2. $d(K(A), \{x\}) \sqsubseteq_2 d(K(B), \{x\}) \iff d(K(A), \{x\}) \sqcap_2 d(K(B), \{x\}) = d(K(A), \{x\})$ or $d(K(A), \{x\}) \sqcup_2 d(K(B), \{x\}) = d(K(B), \{x\})$;
3. $\sum_{x \in U} d(K(A), \{x\}) \sqsubseteq_3 \sum_{x \in U} d(K(B), \{x\}) \iff \sum_{x \in U} d(K(A), \{x\}) \sqcap_3 \sum_{x \in U} d(K(B), \{x\}) = \sum_{x \in U} d(K(A), \{x\})$ or $\sum_{x \in U} d(K(A), \{x\}) \sqcup_3 \sum_{x \in U} d(K(B), \{x\}) = \sum_{x \in U} d(K(B), \{x\})$.

In the following, let us consider the relationships between partial orders \sqsubseteq_i and three orders \preceq_i we defined in Section 2 where $i = 1, 2, 3$.

Theorem 3.4. Let U be the universe of discourse, then we have

1. $K(A) \preceq_1 K(B) \iff K(A) \sqsubseteq_1 K(B)$;
2. $K(A) \preceq_2 K(B) \iff d(K(A), \{x\}) \sqsubseteq_2 d(K(B), \{x\})$;
3. $K(A) \preceq_3 K(B) \iff \sum_{x \in U} d(K(A), \{x\}) \sqsubseteq_3 \sum_{x \in U} d(K(B), \{x\})$.

Proof:

The proof of 1 has been shown in Ref. [14]. In the following, we only prove 2, the proof of 3 is similar to the proof of 2.

$$\forall K(A), K(B) \in K(U),$$

$$\begin{aligned} K(A) \preceq_2 K(B) &\iff |[x]_A| \leq |[x]_B|, \forall x \in U \\ &\iff d([x]_A, \{x\}) \leq d([x]_B, \{x\}), \forall x \in U \\ &\iff \{\min\{d([x]_A, \{x\}), d([x]_B, \{x\})\} : x \in U\} = \{d([x]_A, \{x\}) : x \in U\} \\ &\quad \text{or } \{\max\{d([x]_A, \{x\}), d([x]_B, \{x\})\} : x \in U\} = \{d([x]_B, \{x\}) : x \in U\} \\ &\iff d(K(A), \{x\}) \sqcap_2 d(K(B), \{x\}) = d(K(A), \{x\}) \\ &\quad \text{or } d(K(A), \{x\}) \sqcup_2 d(K(B), \{x\}) = d(K(B), \{x\}). \end{aligned}$$

□

Theorem 3.4 tells us that the three orders we used to judge whether a granulation structure is finer than another one are corresponding to the three partial orders we derived from the proposed algebraic lattices, respectively. In other words, set distance provides us a new approach to study the hierarchies on granulation structures.

Remark 3.5. Though what have been discussed above are all based on the frame of reference the set with single element x , the set with all elements, i.e. U can also be considered as the frame of reference for the constructing of set distances lattices.

4. Algebraic lattices of knowledge distances

In this section, knowledge distance will be used to construct lattices for deriving partial orders. Similar to the construction of set distances lattices, to achieve this goal, we still need a frame of reference. In this paper, the finest granulation structure, i.e. ω is selected. Suppose that $K(U)$ is the set of all the granulation structures in universe U , the $D(K(U), \omega)$ is used to denote the set of knowledge distances between granulation structures in $K(U)$ and ω , i.e.

$$\begin{aligned} D(K(U), \omega) &= \{D(K(A), \omega) : K(A) \in K(U)\} \\ &= \left\{ \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus \{x\}|}{|U|} : K(A) \in K(U) \right\}. \end{aligned}$$

Definition 4.1. Let U be the universe of discourse, $\forall D(K(A), \omega), D(K(B), \omega) \in D(K(U), \omega)$, we may define the following operations:

$$D(K(A), \omega) \wedge_1 D(K(B), \omega) = \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap [x]_B) \oplus \{x\}|}{|U|}; \quad (11)$$

$$D(K(A), \omega) \vee_1 D(K(B), \omega) = \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cup [x]_B) \oplus \{x\}|}{|U|}; \quad (12)$$

$$D(K(A), \omega) \wedge_2 D(K(B), \omega) = \frac{1}{|U|} \sum_{x \in U} \frac{\min\{|[x]_A \oplus \{x\}|, |[x]_B \oplus \{x\}|\}}{|U|}; \quad (13)$$

$$D(K(A), \omega) \vee_2 D(K(B), \omega) = \frac{1}{|U|} \sum_{x \in U} \frac{\max\{|[x]_A \oplus \{x\}|, |[x]_B \oplus \{x\}|\}}{|U|}; \quad (14)$$

$$D(K(A), \omega) \wedge_3 D(K(B), \omega) = \min\left\{\frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus \{x\}|}{|U|}, \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_B \oplus \{x\}|}{|U|}\right\}; \quad (15)$$

$$D(K(A), \omega) \vee_3 D(K(B), \omega) = \max\left\{\frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus \{x\}|}{|U|}, \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_B \oplus \{x\}|}{|U|}\right\}. \quad (16)$$

Through the operations in Definition 4.1, we can further study the algebraic structures of $D(K(U), \omega)$.

Theorem 4.2. Let U be the universe of discourse,

1. $(D(K(U), \omega), \wedge_1, \vee_1)$ is a lattice;
2. $(D(K(U), \omega), \wedge_2, \vee_2)$ is a lattice;
3. $(D(K(U), \omega), \wedge_3, \vee_3)$ is a lattice;

the above three algebraic lattices are referred to as the knowledge distances lattices.

Proof:

We only prove 1, the proofs of 2 and 3 are similar to the proof of 1.

1. idempotent: $\forall D(K(A), \omega) \in D(K(U), \omega)$, by Definition 4.1, we have

$$\begin{aligned} D(K(A), \omega) \wedge_1 D(K(A), \omega) &= \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap [x]_A) \oplus \{x\}|}{|U|} \\ &= \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus \{x\}|}{|U|} \\ &= D(K(A), \omega). \end{aligned}$$

Similarity, it is not difficult to prove that $D(K(A), \omega) \vee_1 D(K(A), \omega) = D(K(A), \omega)$.

2. commutative: $\forall D(K(A), \omega), D(K(B), \omega) \in D(K(U), \omega)$, by Definition 4.1, we have

$$\begin{aligned} D(K(A), \omega) \wedge_1 D(K(B), \omega) &= \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap [x]_B) \oplus \{x\}|}{|U|} \\ &= \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_B \cap [x]_A) \oplus \{x\}|}{|U|} \\ &= D(K(B), \omega) \wedge_1 D(K(A), \omega). \end{aligned}$$

Similarity, it is not difficult to prove that $D(K(A), \omega) \vee_1 D(K(B), \omega) = D(K(B), \omega) \vee_1 D(K(A), \omega)$.

3. associative: $\forall D(K(A), \omega), D(K(B), \omega), D(K(C), \omega) \in D(K(U), \omega)$, by Definition 4.1, we have

$$\begin{aligned} &D(K(A), \omega) \wedge_1 (D(K(B), \omega) \wedge_1 D(K(C), \omega)) \\ &= D(K(A), \omega) \wedge_1 \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_B \cap [x]_C) \oplus \{x\}|}{|U|} \\ &= \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap ([x]_B \cap [x]_C)) \oplus \{x\}|}{|U|} \\ &= \frac{1}{|U|} \sum_{x \in U} \frac{|((([x]_A \cap [x]_B) \cap [x]_C) \oplus \{x\})|}{|U|} \\ &= \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap [x]_B) \oplus \{x\}|}{|U|} \wedge_1 D(K(C), \omega) \\ &= (D(K(A), \omega) \wedge_1 D(K(B), \omega)) \wedge_1 D(K(C), \omega). \end{aligned}$$

Similarity, it is not difficult to prove that $D(K(A), \omega) \vee_1 (D(K(B), \omega) \vee_1 D(K(C), \omega)) = (D(K(A), \omega) \vee_1 D(K(B), \omega)) \vee_1 D(K(C), \omega)$.

4. absorption: $\forall D(K(A), \omega), D(K(B), \omega) \in D(K(U), \omega)$, by Definition 4.1, we have

$$\begin{aligned} D(K(A), \omega) \wedge_1 (D(K(A), \omega) \vee_1 D(K(B), \omega)) &= \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \cap ([x]_A \cup [x]_B) \oplus \{x\}|}{|U|} \\ &= \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus \{x\}|}{|U|} \\ &= D(K(A), \omega). \end{aligned}$$

Similarity, it is not difficult to prove that $D(K(A), \omega) \vee_1 (D(K(A), \omega) \wedge_1 D(K(B), \omega)) = D(K(A), \omega)$. \square

Theorem 4.3. Let U be the universe of discourse,

1. $(D(K(U), \omega), \wedge_1, \vee_1)$ is a distributive lattice;
2. $(D(K(U), \omega), \wedge_2, \vee_2)$ is a distributive lattice;
3. $(D(K(U), \omega), \wedge_3, \vee_3)$ is a distributive lattice.

Proof:

We only prove 1, the proofs of 2 and 3 are similar to the proof of 1.

$\forall D(K(A), \omega), D(K(B), \omega), D(K(C), \omega) \in D(K(U), \omega)$, by Definition 4.1, we have

$$\begin{aligned}
& D(K(A), \omega) \wedge_1 (D(K(B), \omega) \vee_1 D(K(C), \omega)) \\
= & D(K(A), \omega) \wedge_1 \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_B \cup [x]_C) \oplus \{x\}|}{|U|} \\
= & \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap ([x]_B \cup [x]_C)) \oplus \{x\}|}{|U|} \\
= & \frac{1}{|U|} \sum_{x \in U} \frac{|(([x]_A \cap [x]_B) \cup ([x]_A \cup [x]_C)) \oplus \{x\}|}{|U|} \\
= & \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap [x]_B) \oplus \{x\}|}{|U|} \vee_1 \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap [x]_C) \oplus \{x\}|}{|U|} \\
= & \left(\frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus \{x\}|}{|U|} \wedge_1 \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_B \oplus \{x\}|}{|U|} \right) \vee_1 \left(\frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus \{x\}|}{|U|} \wedge_1 \right. \\
& \left. \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_C \oplus \{x\}|}{|U|} \right) \\
= & (D(K(A), \omega) \wedge_1 D(K(B), \omega)) \vee_1 (D(K(A), \omega) \wedge_1 D(K(C), \omega)).
\end{aligned}$$

Similarity, it is not difficult to prove that $D(K(A), \omega) \vee_1 (D(K(B), \omega) \wedge_1 D(K(C), \omega)) = (D(K(A), \omega) \vee_1 D(K(B), \omega)) \wedge_1 (D(K(A), \omega) \vee_1 D(K(C), \omega))$. \square

Similar to the approach we showed in Section 3, through the three algebraic lattices we constructed in Theorem 4.2, three partial orders can be obtained such that:

1. $D(K(A), \omega) \preceq_1 D(K(B), \omega) \iff D(K(A), \omega) \wedge_1 D(K(B), \omega) = D(K(A), \omega) \text{ or } D(K(A), \omega) \vee_1 D(K(B), \omega) = D(K(B), \omega)$;
2. $D(K(A), \omega) \preceq_2 D(K(B), \omega) \iff D(K(A), \omega) \wedge_2 D(K(B), \omega) = D(K(A), \omega) \text{ or } D(K(A), \omega) \vee_2 D(K(B), \omega) = D(K(B), \omega)$;
3. $D(K(A), \omega) \preceq_3 D(K(B), \omega) \iff D(K(A), \omega) \wedge_3 D(K(B), \omega) = D(K(A), \omega) \text{ or } D(K(A), \omega) \vee_3 D(K(B), \omega) = D(K(B), \omega)$;

where $\forall D(K(A), \omega), D(K(B), \omega) \in D(U, \omega)$.

Similar to what have been discussed in Section 3, let us consider the relationships between partial orders \preceq_i and orders \preceq_i where $i = 1, 2, 3$.

Theorem 4.4. Let U be the universe of discourse,

1. $K(A) \preceq_1 K(B) \iff D(K(A), \omega) \trianglelefteq_1 D(K(B), \omega)$;
2. $K(A) \preceq_2 K(B) \iff D(K(A), \omega) \trianglelefteq_2 D(K(B), \omega)$;
3. $K(A) \preceq_3 K(B) \iff D(K(A), \omega) \trianglelefteq_3 D(K(B), \omega)$.

Proof:

We only prove 1, the proofs of 2 and 3 are similar to the proof of 1.

$\forall K(A), K(B) \in K(U)$, we have

$$\begin{aligned}
K(A) \preceq_1 K(B) &\iff [x]_A \subseteq [x]_B, \forall x \in U \\
&\iff ([x]_A \cap [x]_B) \oplus \{x\} = [x]_A \oplus \{x\} \\
&\quad \text{or } ([x]_A \cup [x]_B) \oplus \{x\} = [x]_B \oplus \{x\} \\
&\iff \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cap [x]_B) \oplus \{x\}|}{|U|} = \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_A \oplus \{x\}|}{|U|} \\
&\quad \text{or } \frac{1}{|U|} \sum_{x \in U} \frac{|([x]_A \cup [x]_B) \oplus \{x\}|}{|U|} = \frac{1}{|U|} \sum_{x \in U} \frac{|[x]_B \oplus \{x\}|}{|U|} \\
&\iff D(K(A), \omega) \wedge_1 D(K(B), \omega) = D(K(A), \omega) \\
&\quad \text{or } D(K(A), \omega) \vee_1 D(K(B), \omega) = D(K(B), \omega) \\
&\iff D(K(A), \omega) \trianglelefteq_1 D(K(B), \omega). \quad \square
\end{aligned}$$

Theorem 4.4 tells us that the three orders we used to judge whether a granulation structure is finer than another one are also corresponding to the three partial orders we derived from the proposed knowledge distances lattices, respectively. In other words, knowledge distance also provides us a new approach to study the hierarchies on granulation structures.

Remark 4.5. Though what have been discussed above are all based on the frame of reference the finest granulation structure ω , the coarsest granulation structure, i.e. σ can also be considered as the frame of reference for the constructing of knowledge distances lattices.

5. Conclusions

In this paper, both set distance and knowledge distance are introduced to unravel the hierarchies on granulation structures. In set distance approach, the set with single element is used as the frame of reference while in knowledge distance approach, the finest granulation structure is used as the frame of reference. Based on such two frames of references, three algebraic lattices are constructed, respectively, from which the derived partial orders are corresponding to three different hierarchies on granulation structures. From the viewpoint of distance, these results will provide a more comprehensible perspective for the study of hierarchies.

What have been discussed in this paper is based on the simple granulation structures, which are based on the crisp binary relations, our approach can also be generalized into fuzzy granulation structures and multi-granulation structures to unravel hierarchies. These topics will be addressed in our further research.

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