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## International Journal of Approximate Reasoning

journal homepage: [www.elsevier.com/locate/ijar](http://www.elsevier.com/locate/ijar)NMGRS: Neighborhood-based multigranulation rough sets<sup>☆</sup>Guoping Lin<sup>a,\*</sup>, Yuhua Qian<sup>b</sup>, Jinjin Li<sup>a</sup><sup>a</sup> Department of Mathematics and Information Science, Zhangzhou Normal University, Zhangzhou, 363000 Fujian, China<sup>b</sup> Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Shanxi University, Taiyuan, 030006 Shanxi, China

## ARTICLE INFO

## Article history:

Received 17 May 2011

Received in revised form 3 May 2012

Accepted 24 May 2012

Available online xxx

## Keywords:

Rough sets

Granular computing

Attribute reduction

Multigranulation

Neighborhood relation

## ABSTRACT

Recently, a multigranulation rough set (MGRS) has become a new direction in rough set theory, which is based on multiple binary relations on the universe. However, it is worth noticing that the original MGRS can not be used to discover knowledge from information systems with various domains of attributes. In order to extend the theory of MGRS, the objective of this study is to develop a so-called neighborhood-based multigranulation rough set (NMGRS) in the framework of multigranulation rough sets. Furthermore, by using two different approximating strategies, i.e., seeking common reserving difference and seeking common rejecting difference, we first present optimistic and pessimistic 1-type neighborhood-based multigranulation rough sets and optimistic and pessimistic 2-type neighborhood-based multigranulation rough sets, respectively. Through analyzing several important properties of neighborhood-based multigranulation rough sets, we find that the new rough sets degenerate to the original MGRS when the size of neighborhood equals zero. To obtain covering reducts under neighborhood-based multigranulation rough sets, we then propose a new definition of covering reduct to describe the smallest attribute subset that preserves the consistency of the neighborhood decision system, which can be calculated by Chen's discernibility matrix approach. These results show that the proposed NMGRS largely extends the theory and application of classical MGRS in the context of multiple granulations.

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## 1. Introduction

Rough set theory was originally introduced by Pawlak as a tool to deal with vague, uncertain and incomplete data. It has been found applicable in knowledge discovery, decision analysis, conflict analysis and pattern recognition. One of the applications of rough set theory is to obtain a concept approximation of a universe by two definable subsets called lower and upper approximations. It has been known that lower and upper approximation operators in Pawlak's rough set are defined by an equivalence (indiscernibility) relation [24,25]. With respect to different requirements, in the past ten years, various extensions of Pawlak's rough set have been developed. There are two main methods to generalize it. One method is to extend an equivalence relation to other binary relations, such as a similarity relation, a tolerance relation, and dominance relation [2–5,21–23,26,31,32,34,35,37–40,42,43,51,54–56]. The other is to replace a partition of the universe with a covering and obtained the covering rough sets [1,19,57–59]. Particularly, in order to deal with an information system with numerical attribute, Lin [13–17] presented the neighborhood-based rough set in the neighborhood information system which was originated by Sierpinski and Krieger [36]. Yao studied the neighborhood information system and proposed an approximation retrieval model based on it [49]. Furthermore, Hu et al. [6–9] introduced a different neighborhood-based rough set for heterogeneous feature selection, which can be used to deal with an information system with heterogeneous attributes including categorical attributes and numerical attributes.

<sup>☆</sup> This is an extended version of the paper presented at the 3rd National Conference on Intelligent Information Processing August 12–14, 2011, Shanxi, China.

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43 From above, however, we can find that all extensional rough sets including neighborhood rough sets are constructed on  
 44 the basis of a single binary relation, which limit the applications of rough set theory. In the view of granular computing, they  
 45 are constructed on a single granulation. Accordingly, Qian et al. [28,29] proposed multigranulation rough set in complete  
 46 information system according to a user's different requirements or targets of problem solving. One of important contributions  
 47 in MGRS is to describe the lower and upper approximations of the rough set by multiple equivalence relations (multiple  
 48 granulations) instead of a single equivalence relation (a single granulation). In their papers, Qian et al. said that the MGRS  
 49 are useful in the following cases [28]:

- 50 1. We cannot perform the intersection operations between their quotient sets and the target concept cannot be approx-  
 51 imated by using  $U/(P \cup Q)$  which is called a single granulation in those papers.
- 52 2. In the process of some decision making, the decision or the view of each of decision makers may be independent  
 53 for the same project (or a sample, object and element) in the universe. In this situation, the intersection operations  
 54 between any two quotient sets will be redundant for decision making.
- 55 3. Extract decision rules from distributive information systems and groups of intelligent agents through using rough set  
 56 approaches.

57 Since then, many researchers have extended the classical MGRS by using various generalized binary relations. For instance,  
 58 Qian et al. [29] presented a multigranulation rough set based on multiple tolerance relations in incomplete information  
 59 systems. Lin et al. [18] proposed a covering-based pessimistic multigranulation rough set, Xu et al. [45] proposed another  
 60 generalized version, called variable precision multigranulation rough set, and Yang et al. [47] proposed a multigranulation  
 61 rough set based on a fuzzy binary relation. In fact, the basic idea of multi-granulation has been also discussed by Khan  
 62 et al. in Ref. [11]. However, the existing multigranulation rough set theory can not be used to describe the inconsistency  
 63 coming from a neighborhood information system which consists of numerical and categorical attributes. In order to deal with  
 64 multi-granulation information with heterogeneous attributes, it is necessary to introduce multiple neighborhood relations  
 65 into a neighborhood information system, and further develop a so-called neighborhood-based multigranulation rough sets  
 66 (NMGRS). In particular, we will present two types of neighborhood multigranulation rough sets, 1-type NMGRS and 2-  
 67 type NMGRS. For each NMGRS, we investigate its optimistic version and pessimistic version, respectively, and discuss their  
 68 properties. In addition, we also given a new definition of covering reducts and propose its calculating method, which is based  
 69 on a discernibility matrix approach proposed in the literature [1].

70 The paper is organized as follows. In Section 2, we briefly reviewed some basic concepts of MGRS. In Section 3, a rough set  
 71 based on multi neighborhood relations is presented, called the neighborhood-based multigranulation rough sets (NMGRS),  
 72 and some of its important properties are investigated. In Section 4, we first introduce a concept of covering reduct of the  
 73 neighborhood-based multigranulation rough sets and then employ Chen's discernibility matrix to reduce attributes in the  
 74 neighborhood-based multigranulation rough sets. Finally, Section 5 concludes this study.

## 75 2. Preliminary knowledge on rough sets

76 In this section, we review some basic concepts, which includes Pawlak's rough set, multigranulation rough sets, and  
 77 neighborhood-based rough sets (see [8,13,24,28]).

### 78 2.1. Pawlak's rough set

79 In many data analysis applications, knowledge and information presentation in rough set theory are realized by an  
 80 information system. An information system is a tuple:  $S = (U, AT, \{V_a | a \in AT\}, \{f_a | a \in AT\})$ , where  $U$  is a finite nonempty  
 81 set of objects,  $AT$  is a finite nonempty set of attributes,  $V_a$  is a nonempty set of values of  $a \in AT$ , and  $f_a: U \rightarrow V_a$  is an  
 82 information function that maps an object in  $U$  to exactly one value in  $V_a$ .

83 In particular, a target information system is given by  $S = (U, AT \cup D, \{V_a | a \in AT\}, \{f_a | a \in AT\})$ , where  $AT$  is a set  
 84 of condition attributes describing the objects, and  $D$  is a set of decision attributes that indicate the classes of objects. In  
 85 general, we often consider the decision information system with only one decision attribute, because an information system  
 86 with multi decision attributes can be easily transformed into a system with a single decision attribute by considering the  
 87 Cartesian product of the original decision attributes [35,50].

88 Each nonempty subset  $B \subseteq AT$  determines an indiscernibility relation, defined as  $R_B = \{(x, y) \in U \times U \mid f_a(x) =$   
 89  $f_a(y), \forall a \in B\}$ .

90 The relation  $R_B$  partitions  $U$  into some equivalence classes given by  $U/R_B = \{[x]_B \mid x \in U\}$ , where  $[x]_B = \{y \in U \mid (x, y) \in$   
 91  $R_B\}$ .

92 For  $X \subseteq U$ , sets  $\underline{R}_B X = \cup\{Y \in U/IND(B) \mid Y \subseteq X\}$  and  $\overline{R}_B X = \cup\{Y \in U/IND(B) \mid Y \cap X \neq \emptyset\}$  are called the lower and  
 93 the upper approximations of  $X$  with respect to  $B$ , respectively.

The area of uncertainty or boundary region is

$$Bn(X) = \overline{R}_B X \setminus \underline{R}_B X.$$

94 In order to measure the imprecision of a rough set, Pawlak [25] recommended for  $X \neq \emptyset$ , the ratio  $\alpha_{R_B}(X) = \frac{|R_B X|}{|R_B X|}$ , which is  
 95 called the accuracy measure of  $X$  by  $R_B$ . Roughness is calculated by subtracting the accuracy from  $\alpha_{R_B}$ :  $\rho_{R_B}(X) = 1 - \alpha_{R_B}(X)$ .

96 2.2. Multigranulation rough sets (MGRS)

97 In recent years, Qian et al. [28] have proposed a new extension of Pawlak rough set, i.e., multigranulation rough sets  
 98 (MGRS). In the multigranulation rough set theory, a target concept is approximated by multiple binary relations. Furthermore,  
 99 two kinds of important multigranulation rough sets were presented with optimistic and pessimistic strategies, which are  
 100 called optimistic multigranulation rough sets and pessimistic multigranulation rough sets, respectively [28,30].

**Definition 1.** Let  $S = (U, AT, f)$  be an information system,  $A_1, A_2, \dots, A_m \subseteq AT$ , and  $X \subseteq U$ . The optimistic lower approximation and the upper approximation of  $X$  with respect to  $A_1, A_2, \dots, A_m$  are denoted by  $\sum_{i=1}^m A_i^O X$  and  $\overline{\sum_{i=1}^m A_i^O X}$ , respectively, where

$$\sum_{i=1}^m A_i^O X = \bigcup \{x \in U \mid [x]_{A_i} \subseteq X, \text{ for some } i \leq m\}, \tag{1}$$

$$\overline{\sum_{i=1}^m A_i^O X} = \sim \sum_{i=1}^m A_i^O (\sim X). \tag{2}$$

101 Then  $(\sum_{i=1}^m A_i^O X, \overline{\sum_{i=1}^m A_i^O X})$  is the optimistic MGRS [24]. The word “optimistic” is used to express the idea that in multiple  
 102 independent granular structures, one needs only at least one granular structure to satisfy with the inclusion condition  
 103 between an equivalence class and a target concept. The upper approximation of the optimistic multigranulation rough set  
 104 is defined by the complement of the lower approximation.

And the area of uncertainty or boundary region in MGRS is

$$Bn_{\sum_{i=1}^m A_i^O}^O(X) = \overline{\sum_{i=1}^m A_i^O X} \setminus \sum_{i=1}^m A_i^O X.$$

The definition of pessimistic MGRS [30] is defined as follows:

$$\sum_{i=1}^m A_i^P(X) = \{x \in U \mid [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \dots \wedge [x]_{A_m} \subseteq X\}, \tag{3}$$

$$\overline{\sum_{i=1}^m A_i^P(X)} = \sim \sum_{i=1}^m A_i^P(\sim X). \tag{4}$$

Then  $(\sum_{i=1}^m A_i^P X, \overline{\sum_{i=1}^m A_i^P X})$  is the pessimistic MGRS [30]. The word “pessimistic” is used to express the idea that in  
 multiple independent granular structures, one needs all granular structures to satisfy with the inclusion condition  
 between an equivalence class and a target concept. The upper approximation of the optimistic multigranulation rough set is also  
 defined by the complement of the lower approximation. And the area of uncertainty or boundary region in MGRS is

$$BN_{\sum_{i=1}^m A_i^P}^P(X) = \overline{\sum_{i=1}^m A_i^P(X)} \setminus \sum_{i=1}^m A_i^P(X).$$

105 2.3. Neighborhood-based rough sets

106 In order to make Pawlak’s rough set deal with the information system with heterogeneous attributes, T. Y. Lin et al.  
 107 [14] gave the concept of neighborhood and proposed neighborhood-based rough sets. Since then, many researchers further  
 108 studied the theory of the neighborhood-based rough set [6–10, 15, 48]. In this section, we especially introduce some concepts  
 109 of neighborhood-based rough sets proposed by Hu [8].

110 **Definition 2.** Let  $S = (U, AT, f)$  be an information system with heterogeneous attributes,  $X \subseteq U$  and  $A, B \subseteq AT$  are  
 111 categorical and numerical attributes, respectively. The neighborhood granules of objects  $x$  induced by  $A, B, A \cup B$  are defined  
 112 as

**Table 1**  
A target information system with heterogeneous attributes.

	Outlook	Ultra-ray	Temperature	Humidity	Windy	Intensity	Play
$x_1$	Sunny	Weak	85	85	False	85	No
$x_2$	Sunny	Strong	80	90	True	95	No
$x_3$	Overcast	Strong	86	85	False	82	Yes
$x_4$	Rainy	Middle	70	96	False	91	Yes
$x_5$	Rainy	Middle	68	80	False	80	Yes
$x_6$	Rainy	Weak	65	70	True	75	No
$x_7$	Overcast	Middle	64	65	True	63	Yes
$x_8$	Sunny	Strong	72	95	False	90	No

- 113 (1)  $n_A(x) = \{x_i \in U \mid d_A(x, x_i) = 0\}$ ;  
 114 (2)  $n_B(x) = \{x_i \in U \mid d_B(x, x_i) \leq \delta\}$ ;  
 115 (3)  $n_{(A \cup B)}(x) = \{x_i \in U \mid d_A(x, x_i) = 0 \wedge d_B(x, x_i) \leq \delta\}$ ,

116 where  $d$  is a distance [40] between  $x$  and  $y$ ,  $\delta$  is a nonnegative number, and “ $\wedge$ ” means “and” operator. (1) is designed for  
 117 numerical attributes; (2) is designed for categorical attributes, and (3) is designed for heterogeneous attributes, namely,  
 118 categorical and numerical attributes.

A neighborhood relation  $N$  on the universe can be written as a relation matrix  $M(N) = (r_{ij})_{n \times n}$ , where

$$r_{ij} = \begin{cases} 1, & d(x_i, x_j) \leq \delta, \\ 0, & \text{otherwise.} \end{cases}$$

119 Accordingly, we say  $(U, N)$  a neighborhood approximation space. If there is an attribute subset in the system generating  
 120 a neighborhood relation on the universe, we can regard this system as a neighborhood information system, denoted by  
 121  $NIS = (U, AT, N)$ , where  $U$  is a nonempty finite set and  $AT$  is an attribute set. In particular, a neighborhood information system  
 122 is also called a neighborhood decision information system if we distinguish condition attributes and decision attributes,  
 123 denoted by  $NIS = (U, AT \cup D, N)$ .

124 **Example 1.** Here, we use an example to illustrate some notions of an information system which consists of categorical and  
 125 numerical attributes. Table 1 shows data set *play tennis* with heterogeneous attributes, namely, categorical and numerical  
 126 attributes, where  $U = \{x_1, x_2, \dots, x_8\}$ ,  $AT = \{\text{outlook, ultra-ray, temperature, humidity, intensity, windy}\}$ ,  $D = \{\text{play}\}$ . Espe-  
 127 cially, *Outlook, ultra-ray*, and *windy* are categorical condition attributes, *temperature, humidity* and *intensity* are numerical  
 128 condition attributes, and *play* is a decision attribute. In the sequel,  $O, U, T, H, W, I$  will displace *outlook, ultra-ray, temper-*  
 129 *ature, humidity, windy*, and *intensity*, respectively. In Table 1, in order to reduce sample classification error rate caused by  
 130 inconsistent dimension, numerical attribute values are standardized into  $[0, 1]$  for computing, see [7].

**Definition 3.** Let  $(U, N)$  be a neighborhood approximation space. For any  $X \subseteq U$ , the lower approximation and upper  
 approximation of  $X$  in  $U$  are defined as:

$$\underline{NX} = \{x \in U \mid n(x) \subseteq X\}, \tag{5}$$

$$\overline{NX} = \{x \in U \mid n(x) \cap X \neq \emptyset\}. \tag{6}$$

131 One calls  $(\underline{NX}, \overline{NX})$  a neighborhood rough set. Obviously,  $\underline{NX} \subseteq X \subseteq \overline{NX}$ . The *boundary region* of  $X$  in the approximation  
 132 space is defined as  $Bn(X) = \overline{NX} \setminus \underline{NX}$ .

133 The size of boundary region reflects the degree of roughness of set  $X$  in the neighborhood approximation space  $(U, N)$ .  
 134 In the neighborhood rough set,  $\delta$  can be considered as a parameter to control the granularity level at which we analyze the  
 135 classification task.

### 136 3. Neighborhood multigranulation rough sets

137 In this section, we extend the classical MGRS to neighborhood-based multigranulation rough sets (NMGRS). We propose  
 138 two types of neighborhood multigranulation rough sets according to different representations of neighborhood information  
 139 granules by Definition 3. In the first case, a granular space induced by a neighborhood relation on the universe can be  
 140 regarded as a set of mixed information granules induced by both a similarity relation and an indiscernibility relation in  
 141 the view of granular computing [53]. If the approximations of a target concept are described by these mixed information  
 142 granules, we call this rough set a 1-type neighborhood multigranulation rough set in this paper, denoted by 1-type NMGRS.  
 143 In the second case, if the approximations of a target concept are described by multiple neighborhood relations, we call this  
 144 rough set a 2-type neighborhood multigranulation rough set, denoted by 2-type NMGRS.

145 In the following, we will give the definitions of optimistic 1-type NMGRS and optimistic 2-type NMGRS and the defin-  
146 itions of pessimistic versions, respectively. Conveniently, we mainly discuss the properties of the optimistic versions. The  
147 pessimistic versions can be done similarly. We hence omit them in this paper.

148 3.1. 1-type neighborhood multigranulation rough sets (1-type NMGRS)

149 As we know, the incomplete MGRS is based on multiple tolerance relations, which sometimes can be also regarded  
150 as a neighborhood relation [7]. However, these existing multigranulation versions still can not deal with data sets with  
151 heterogeneous attributes. Therefore, it is necessary to develop a new rough set based on multiple neighborhood relations  
152 to deal with hybrid data. Simply, we first investigate the approximation of a target set induced by mixed granules on the  
153 universe, which can be regarded as a simple neighborhood multigranulation rough set, just 1-type NMGRS.

**Definition 4** (1-type NMGRS). Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A \subseteq AT$  a categorical attribute set,  $B \subseteq AT$  a numerical attribute set,  $A \cup B \subseteq AT$  a mixed attribute set;  $U/A, U/B$ , and  $U/(A \cup B)$  represent a partition and two coverings of the universe  $U$ , respectively. For any  $X \subseteq U$ , the optimistic multigranulation lower and upper approximations of  $X$  with respect to  $A, B$  in  $U$  are defined in the following:

$$(A + B)^0 X = \{x \in U \mid n_A(x) \subseteq X \vee n_B(x) \subseteq X\}, \tag{7}$$

$$\overline{(A + B)^0 X} = \sim (A + B)^0 (\sim X). \tag{8}$$

By Definition 4, we can see that the lower and upper approximations of  $X$  of optimistic 1-type NMGRS satisfy duality property, i.e., the upper approximation can be defined by the complement of the lower approximation. The *area of uncertainty or boundary region* is defined as

$$Bn_{(A+B)}^0(X) = \overline{(A + B)^0 X} \setminus (A + B)^0 X.$$

154 We call  $((A + B)^0 X, \overline{(A + B)^0 X})$  an optimistic 1-type NMGRS. Obviously, the optimistic 1-type NMGRS can degenerate into  
155 the original optimistic multigranulation while  $\delta = 0$ . The original MGRS is a special instance of 1-type NMGRS.

156 **Theorem 1.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A, B \subseteq AT$  categorical and numerical attribute subsets,  
157 respectively. For any  $X \subseteq U$ , then

$$158 \overline{(A + B)^0 X} = \{x \in U \mid (n_A(x) \cap X \neq \emptyset) \wedge (n_B(x) \cap X \neq \emptyset)\}.$$

159 **Proof.** By Definition 4, we have that

$$\begin{aligned} 160 x \in \overline{(A + B)^0 X} &\Leftrightarrow x \in \sim (A + B)^0 (\sim X) \\ 161 &\Leftrightarrow x \notin (A + B)^0 (\sim X) \\ 162 &\Leftrightarrow n_A(x) \not\subseteq (\sim X) \wedge n_B(x) \not\subseteq (\sim X) \\ 163 &\Leftrightarrow n_A(x) \cap X \neq \emptyset \wedge n_B(x) \cap X \neq \emptyset. \end{aligned}$$

164 This completes the proof.  $\square$

165 By Theorem 1, we can see that though the optimistic multigranulation upper approximation is defined by the complement  
166 of the optimistic multigranulation lower approximation, it also can be constructed by objects with nonempty intersection  
167 with the target concept in terms of each granular structure.

168 **Proposition 1.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $\forall A, B \subseteq AT$ , and  $\forall X \subseteq U$ , then

$$169 (1) (A + B)^0 X = \underline{AX} \cup \underline{BX},$$

$$170 (2) \overline{(A + B)^0 X} = \overline{AX} \cap \overline{BX}.$$

171 **Proof.** (1) Let  $x \in \underline{AX}$  ( $x \in U$ ), note that  $\underline{AX} = \{x \in U \mid n_A(x) \subseteq X\}$ , but  $x \in (A + B)^0 X$ , hence  $\underline{AX} \subseteq (A + B)^0 X$ . Similarly,  
172  $\underline{BX} \subseteq (A + B)^0 X$ . So  $(A + B)^0 X \supseteq \underline{AX} \cup \underline{BX}$ . And, for  $x \in (A + B)^0 X$ , from (7), we have either  $n_A(x) \subseteq X$ , then  $x \in \underline{AX}$  or  
173  $n_B(x) \subseteq X$ , then  $x \in \underline{BX}$ , therefore  $x \in \underline{AX} \cup \underline{BX}$ , namely,  $(A + B)^0 X \subseteq \underline{AX} \cup \underline{BX}$ . Therefore,  $(A + B)^0 X = \underline{AX} \cup \underline{BX}$ .

174 (2) From above and (8), we have  $\overline{(A + B)^0 X} = \sim (A + B)^0 (\sim X) = \sim (\underline{A(\sim X)} \cup \underline{B(\sim X)}) = \overline{A(\sim X)} \cap \overline{B(\sim X)}$ .

175 This completes the proof.  $\square$

176 **Corollary 1.**  $Bn_{(A+B)}^0(X) \subseteq Bn_A(X) \cup Bn_B(X)$ .

177 In what follows, we will illuminate the difference between the 1-type NMGRS and classical Pawlak's rough sets through  
178 employing Example 2.



179 **Example 2** (Continued from Example 1). Let  $X = \{x_1, x_2, x_3, x_7\}$ . Here we compute the neighborhood granules of samples  
180 with  $\delta = 0.1$ . A partition and two coverings are induced from Table 1 as follows:

181 Let  $A = \{O, W\} \subseteq AT$  be a categorical attribute subset. According to Definition 2, the information granules induced by  $A$   
182 are listed.  $n_A(x_1) = \{x_1, x_8\} = n_A\{x_8\}$ ,  $n_A(x_2) = \{x_2\}$ ,  $n_A(x_3) = \{x_3\}$ ,  $n_A(x_4) = \{x_4, x_5\} = n_A\{x_5\}$ ,  $n_A(x_6) = \{x_6\}$ ,  $n_A(x_7) =$   
183  $\{x_7\}$ . Obviously, they form a covering of the universe, i.e.,  $U/A = \{\{x_1, x_8\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_5, x_4\}, \{x_6\}, \{x_7\}, \{x_8, x_1\}\}$   
184 which is a granular structure on  $U$ , then  $\underline{A}X = \{x_2, x_3, x_7\}$  and  $\overline{A}X = \{x_1, x_2, x_3, x_7, x_8\}$ .

185 Let  $B = \{T, H\} \subseteq AT$  be a numerical attribute subset. Then, we have that  $n_B(x_1) = \{x_1, x_2, x_3\} = n_B(x_3)$ ,  $n_B(x_2) =$   
186  $\{x_2, x_1, x_3, x_4, x_8\}$ ,  $n_B(x_4) = \{x_4, x_2, x_8\}$ ,  $n_B(x_5) = \{x_5, x_6\}$ ,  $n_B(x_6) = \{x_6, x_5, x_7\}$ ,  $n_B(x_7) = \{x_7, x_6\}$ ,  $n_B(x_8) =$   
187  $\{x_8, x_2, x_4\}$ . Similarly, they form a covering of the universe, i.e.,  $U/B = \{\{x_1, x_2, x_3\}, \{x_2, x_1, x_3, x_4, x_8\}, \{x_3, x_1, x_2\}, \{x_4, x_2,$   
188  $x_8\}, \{x_5, x_6\}, \{x_6, x_5, x_7\}, \{x_7, x_6\}, \{x_8, x_2, x_4\}\}$ . Therefore we have that  $\underline{B}X = \{x_1, x_3\}$ ,  $\overline{B}X = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\}$ .

189 Based on  $U/A$  and  $U/B$  induced by  $A$  and  $B$ , we have the optimistic lower and upper approximations of  $X$  in  $U$ ,  
190 respectively,  $(A+B)^O X = \{x_1, x_2, x_3, x_7\} = \underline{A}(X) \cup \underline{B}(X)$ ,  $(A+B)^O X = \sim (\overline{A+B})(\sim X) = \{x_1, x_2, x_3, x_7, x_8\} =$   
191  $\overline{A}(X) \cap \overline{B}(X)$ .

192 Furthermore, By the term (3) in Definition 2, we have that  $U/(A \cup B) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}, \{x_8\}\}$ .  
193 Obviously,  $U/(A \cup B)$  also forms a covering of the universe  $U$ . Then, we have  $(A \cup B)X = \{x_1, x_2, x_3, x_7\}$ ,  $(A \cup B)X =$   
194  $\{x_1, x_2, x_3, x_7\}$ . Easily,  $(A \cup B)X \supseteq (A+B)^O X$ ,  $(A \cup B)X \subseteq (A+B)^O X$ .

195 As a result of this example, we have the following results.

196 **Proposition 2.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A, B \subseteq AT$  categorical and numerical attribute  
197 subsets, respectively. For any  $X \subseteq U$ , then

- 198 (1)  $(A+B)^O X \subseteq (A \cup B)X$ ,  
199 (2)  $(A+B)^O X \supseteq (A \cup B)X$ .

200 **Proof.** (1) For any  $x \in (A+B)^O X$ , by Definition 4, it follows that  $x \in n_A(x)$  and  $x \in n_B(x)$ . Hence  $x \in n_A(x) \cap n_B(x)$ . But  
201  $n_A(x) \cap n_B(x) \subseteq n_{(A \cup B)}(x)$  for all  $x \in U$ . Therefore,  $x \in (A \cup B)X$ , i.e.  $(A+B)^O X \subseteq (A \cup B)X$ .

202 (2) From Pawlak's rough set theory, we know  $(A \cup B)X = \sim (\overline{A \cup B})(\sim X)$ , applying the result of (1), we have that  
203  $(A \cup B)X \supseteq (A+B)^O(\sim X)$ . Hence,  $\sim (\overline{A \cup B})(\sim X) \subseteq \sim (\overline{A+B})(\sim X)$ , i.e.,  $(A+B)^O X \supseteq (A \cup B)X$ .  
204 This completes the proof.  $\square$

205 Proposition 2 shows that the optimistic lower approximation is not more than the Pawlak's lower approximation, while  
206 the optimistic upper approximation is not less than the Pawlak's upper approximation.

207 **Corollary 2.**  $Bn_{(A+B)}^O(X) \supseteq Bn_{(A \cup B)}(X)$ .

**Corollary 3.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A, B \subseteq AT$  categorical and numerical attribute  
subsets, respectively, and  $X \subseteq U$ . Then

$$\alpha_{(A \cup B)}(X) \geq \alpha_{(A+B)}(X).$$

208 **Proof.** They are straightforward from the definition of accuracy measure of  $X$ .

209 In what follows, we further clarify the difference between multigranulation rough sets and classical rough sets. It can be  
210 illustrated from the following four aspects.

- 211 (1) Multigranulation rough set theory is a strategy for information fusion through single granulation rough sets. Here,  
212 neighborhood-based multigranulation rough sets is a simple information fusion method by operations ' $\vee$ ' (conjunction)  
213 or ' $\wedge$ ' (disjunction).  
214 (2) In fact, there are some other fusion strategies [20,45–47]. For instance, in the literature [45], Xu et al. introduced a  
215 supporting characteristic function and a variable precision parameter  $\beta$  called information level to investigate a target  
216 concept under majority granulations.  
217 (3) It is Proposition 2 that embodies the difference between classic rough sets and multigranulation rough sets.  
218 (4) With regard to some special information systems, such as multi-source information systems, distributive information  
219 systems and groups of intelligent agents, the classical rough sets can not deal with these information systems, but  
220 multigranulation rough sets can.  $\square$

221 **Proposition 3.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A, B \subseteq AT$  categorical and numerical attribute  
subsets, respectively,  $X \subseteq U$ , and  $\delta_1, \delta_2$  two nonnegative numbers. If  $\delta_1 \geq \delta_2$ , then

222 (1)  $\overline{(A + B)}_{\delta_1}^0 X \subseteq \overline{(A + B)}_{\delta_2}^0 X$ ,

223 (2)  $\overline{(A + B)}_{\delta_1}^0 X \supseteq \overline{(A + B)}_{\delta_2}^0 X$ .

224 **Proof.** (1) Let  $X \subseteq U$ , assume that  $\overline{(A + B)}_{\delta}^0 X = \{x \mid n_A^{\delta}(x) \subseteq X \vee n_B^{\delta}(x) \subseteq X\}$ , for any  $x \in U$ . If  $\delta_1 \geq \delta_2$ , we obviously have  
 225  $n_A^{\delta_1}(x) \subseteq n_A^{\delta_2}(x)$  and  $n_B^{\delta_1}(x) \subseteq n_B^{\delta_2}(x)$ . So for any  $x \in n_A^{\delta_1}(x) \subseteq X$ , we have  $x \in n_A^{\delta_2}(x) \subseteq X$ . Similarly, for any  $x \in n_B^{\delta_1}(x) \subseteq X$ ,  
 226 we have  $x \in n_B^{\delta_2}(x) \subseteq X$ . Therefore, we have  $x \in \overline{(A + B)}_{\delta_2}^0 X$  if  $x \in \overline{(A + B)}_{\delta_1}^0 X$ . Hence,  $\overline{(A + B)}_{\delta_1}^0 X \subseteq \overline{(A + B)}_{\delta_2}^0 X$ .

227 (2) Similarly, we can prove that  $\overline{(A + B)}_{\delta_1}^0 X \supseteq \overline{(A + B)}_{\delta_2}^0 X$ .  
 228 This completes the proof.  $\square$

229 Proposition 3 shows that the size of lower approximation of  $X$  under a 1-type optimistic neighborhood-based multigranulation rough set will become much larger with the value of the parameter  $\delta$  being much bigger. Its upper approximation  
 230 has the inverse conclusion.  
 231

232 **Proposition 4.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A, B \subseteq AT$  categorical and numerical attribute  
 233 subsets, respectively, and  $X, Y \subseteq U$ . If  $X \subseteq Y$ , then

234 (1)  $\overline{(A + B)}^0 X \subseteq \overline{(A + B)}^0 Y$ ,

235 (2)  $\overline{(A + B)}^0 X \supseteq \overline{(A + B)}^0 Y$ .

236 **Proof.** (1) If  $X \subseteq Y$ , then  $X \cap Y = X$ . Then we have  
 237  $\overline{(A + B)}^0 X = \overline{(A + B)}^0 (X \cap Y)$   
 238  $= \overline{A}(X \cap Y) \cup \overline{B}(X \cap Y)$   
 239  $= (\overline{A}X \cap \overline{A}Y) \cup (\overline{B}X \cap \overline{B}Y)$   
 240  $= (\overline{A}X \cap \overline{A}Y) \cup \overline{B}X \cap ((\overline{A}X \cap \overline{A}Y) \cup \overline{B}Y)$   
 241  $= (\overline{A}X \cup \overline{B}X) \cap (\overline{A}Y \cup \overline{B}X) \cap (\overline{A}X \cup \overline{B}Y) \cap (\overline{A}Y \cup \overline{B}Y)$   
 242  $= (\overline{(A + B)}^0 X \cap \overline{(A + B)}^0 Y) \cap ((\overline{A}Y \cup \overline{B}X) \cap (\overline{A}X \cup \overline{B}Y))$   
 243  $\subseteq (\overline{(A + B)}^0 X \cap \overline{(A + B)}^0 Y) \subseteq \overline{(A + B)}^0 Y$ .

244 Hence,  $\overline{(A + B)}^0 X \subseteq \overline{(A + B)}^0 Y$ .  
 245 (2) If  $X \subseteq Y$ , then  $X \cup Y = Y$ . Then we have  
 246  $\overline{(A + B)}^0 Y = \overline{(A + B)}^0 (X \cup Y)$   
 247  $= \overline{A}(X \cup Y) \cap \overline{B}(X \cup Y)$   
 248  $= (\overline{A}X \cup \overline{A}Y) \cap (\overline{B}X \cup \overline{B}Y)$   
 249  $= ((\overline{A}X \cup \overline{A}Y) \cap \overline{B}X) \cup ((\overline{A}X \cup \overline{A}Y) \cap \overline{B}Y)$   
 250  $= (\overline{A}X \cap \overline{B}X) \cup (\overline{A}Y \cap \overline{B}X) \cup (\overline{A}X \cap \overline{B}Y) \cup (\overline{A}Y \cap \overline{B}Y)$   
 251  $= \overline{(A + B)}^0 X \cup \overline{(A + B)}^0 Y \cup (\overline{A}X \cap \overline{B}Y) \cup (\overline{A}Y \cap \overline{B}Y)$   
 252  $\supseteq \overline{(A + B)}^0 X \cup \overline{(A + B)}^0 Y \supseteq \overline{(A + B)}^0 X$ .

253 Hence,  $\overline{(A + B)}^0 Y \supseteq \overline{(A + B)}^0 X$ .  
 254 This completes the proof.  $\square$

**Corollary 4.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A, B \subseteq AT$  categorical and numerical attribute  
 subsets, respectively, and  $X \subseteq U$ . If  $\delta_1, \delta_2$  are two nonnegative numbers and  $\delta_1 \geq \delta_2$ , then

$$\alpha_{(A+B)\delta_1}^0(X) \leq \alpha_{(A+B)\delta_2}^0(X).$$

255 **Proof.** It is straightforward from Proposition 3.

Similar to the classical pessimistic MGRS's definition [26], let  $NIS = (U, AT, N)$  be a neighborhood information system,  
 where  $A, B \subseteq AT$  are categorical and numerical attributes, respectively. For any  $X \subseteq U$ , the lower and upper approximations  
 of  $X$  of the pessimistic 1-type NMGRS in  $U$  are described as:

$$\underline{(A + B)}^P X = \{x \in U \mid n_A(x) \subseteq X \wedge n_B(x) \subseteq X\}, \tag{9}$$

$$\overline{(A + B)}^P X = \sim \underline{(A + B)}^P (\sim X). \tag{10}$$

Analogously, this multigranulation boundary region of  $X$  is

$$Bn_{(A+B)}^P(X) = \overline{(A + B)}^P X \setminus \underline{(A + B)}^P X.$$



256 We call  $((A + B)^P X, \overline{(A + B)^P X})$  a pessimistic 1-type neighborhood multigranulation rough set.  $\square$

257 **Theorem 2.** Let  $NIS = (U, AT, N)$  be a neighborhood information system, where  $A, B \subseteq AT$  are categorical and numerical  
258 attributes, respectively. For any  $X \subseteq U$ , then  $\overline{(A + B)^P X} = \{x \in U \mid (n_A(x) \cap X \neq \emptyset) \vee (n_B(x) \cap X \neq \emptyset)\}$ .

259 **Proof.** By the above definitions, we have

$$\begin{aligned} 260 \quad x \in \overline{(A + B)^P X} &\Leftrightarrow x \in \sim (A + B)^P (\sim X) \\ 261 \quad &\Leftrightarrow x \notin (A + B)^P (\sim X) \\ 262 \quad &\Leftrightarrow n_A(x) \not\subseteq (\sim X) \vee n_B(x) \not\subseteq (\sim X) \\ 263 \quad &\Leftrightarrow n_A(x) \cap X \neq \emptyset \vee n_B(x) \cap X \neq \emptyset. \end{aligned}$$

264 This completes the proof.  $\square$

265 Different from the upper approximation of optimistic 1-type neighborhood multigranulation rough set, the upper ap-  
266 proximation of pessimistic 1-type neighborhood multigranulation rough set is represented as a set in which objects have  
267 non-empty intersection with the target in terms of at least one granular structure.

268 From the above analysis, we can obtain the following two corollaries and one proposition.

269 **Corollary 5.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A, B \subseteq AT$  categorical and numerical attributes,  
270 respectively. For any  $X \subseteq U$ , then  $\overline{(A + B)^P X} = \overline{AX} \cup \overline{BX}$ .

271 **Proof.**  $\overline{(A + B)^P X} = \sim (A + B)^P (\sim X)$

$$272 \quad = \sim (\underline{A}(\sim X) \cap \underline{B}(\sim X))$$

$$273 \quad = \sim \underline{A}(\sim X) \cup \sim \underline{B}(\sim X)$$

$$274 \quad = \overline{AX} \cup \overline{BX}.$$

275 This completes the proof.  $\square$

276 Similarly, other properties of the pessimistic version can be proved by the same method.

### 277 3.2. 2-Type neighborhood multigranulation rough sets (2-type NMGRS)

278 When multiple neighborhood relations are used in the neighborhood information system, we call such a multigranulation  
279 rough set a 2-type neighborhood multigranulation rough set, denoted by 2-type NMGRS. Simply, we first investigate how to  
280 approximate a target concept through two neighborhood relations. For simpleness, we use the denotations  $\underline{A + B} X = \underline{NX}$ ,  
281 and  $\overline{A + B} X = \overline{NX}$  in the following:

**Definition 5** (2-type NMGRS). Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2$  two neighborhood  
relations on the universe  $U$ ,  $N_1$  induced by  $A_1$  and  $B_1$ ,  $N_2$  induced by  $A_2$  and  $B_2$ , where  $A_1, A_2$  are two categorical attribute  
subsets and  $B_1, B_2$  are two numerical attribute subsets, and  $U/A_1, U/A_2, U/B_1, U/B_2$  are four coverings on the universe  $U$ .  
Then for any  $X \subseteq U$ , the optimistic lower approximation and upper approximation of  $X$  in  $U$  are defined as

$$(N_1 + N_2)^O X = \{x \in U \mid n_{(A_1+B_1)}(x) \subseteq X \vee n_{(A_2+B_2)}(x) \subseteq X\}, \quad (11)$$

$$\overline{(N_1 + N_2)^O X} = \sim (N_1 + N_2)^O (\sim X). \quad (12)$$

The area of uncertainty or boundary region is defined as:

$$Bn_{(N_1+N_2)}^O(X) = \overline{(N_1 + N_2)^O X} \setminus \underline{(N_1 + N_2)^O X}.$$

282 We call  $((N_1 + N_2)^O X, \overline{(N_1 + N_2)^O X})$  an optimistic 2-type NMGRS based on two neighborhood relations.

283 In 2-type NMGRS,  $n_{(A+B)}(x)$  represents a neighborhood induced by a heterogeneous attribute subset and  $n_{(A+B)}(x) =$   
284  $\{x \in U \mid n_A(x) \leq \delta \vee n_B(x) \leq \delta\}$ . However, by the (3) of Definition 2,  $n_{(A \cup B)}(x) = \{x_i \in U \mid d_A(x, x_i) = 0 \wedge d_B(x, x_i) \leq \delta\}$ .

285 It is deserved to point out that let  $NIS = (U, AT, N)$  be a neighborhood information system, a partition  $U/A$  induced by  
286 a categorical attribute subset  $A$ , and a covering  $U/B$  induced by a numerical attribute subset  $B$ , then  $U/(A \cup B)$  induced by  
287  $A \cup B$  is also a covering of the universe.

288 **Example 3** (Continued from Example 1). Let  $X = \{x_1, x_2, x_3, x_7\}$ , four coverings on the universe  $U$  are induced from  
289 Table 1 as follows:

290 Let  $A_1 = \{O, W\} \subseteq AT$  be a categorical attribute subset, from Example 2, it follows that  $U/A_1 = \{\{x_1, x_8\}, \{x_2\}, \{x_3\},$   
 291  $\{x_4, x_5\}, \{x_6\}, \{x_7\}\}$ . Then  $A_1X = \{x_2, x_3, x_7\}$ ,  $\overline{A_1}X = \{x_1, x_2, x_3, x_7, x_8\}$ .

292 Let  $A_2 = \{O, U\} \subseteq AT$  be a categorical attribute subset, from Table 1, it follows that  $U/A_2 = \{\{x_1\}, \{x_2, x_8\}, \{x_3\},$   
 293  $\{x_4, x_5\}, \{x_6\}, \{x_7\}\}$ . Then  $A_2X = \{x_1, x_3, x_7\}$ ,  $\overline{A_2}X = \{x_1, x_2, x_3, x_7, x_8\}$ .

294 Let  $B_1 = \{T, H\} \subseteq AT$  be numerical attribute subset, from Example 2, it follows that  $U/B_1 = \{\{x_1, x_2, x_3\}, \{x_2, x_1,$   
 295  $x_3, x_4, x_8\}, \{x_3, x_1, x_2\}, \{x_4, x_2, x_8\}, \{x_5, x_6\}, \{x_6, x_5, x_7\}, \{x_7, x_6\}, \{x_8, x_4, x_2\}\}$ , we have that  $B_1X = \{x_1, x_3\}$ ,  $\overline{B_1}X = \{x_1, x_2,$   
 296  $x_3, x_4, x_6, x_7, x_8\}$ .

297 Let  $B_2 = \{T, I\} \subseteq AT$  be a numerical attribute subset, from Table 1, it follows that  $U/B_2 = \{\{x_1, x_2, x_3\}, \{x_2, x_1, x_4, x_8\},$   
 298  $\{x_3, x_1\}, \{x_4, x_2, x_8\}, \{x_5, x_6, x_8\}, \{x_6, x_5\}, \{x_7\}, \{x_8, x_2, x_4, x_5\}\}$ . We have that  $B_2X = \{x_1, x_3, x_7\}$ ,  $\overline{B_2}X = \{x_1, x_2, x_3, x_4, x_7,$   
 299  $x_8\}$ . From the definition of the optimistic 1-type NMGRS, by computing, we have that  $(A_1 + B_1)^{\circ}X = \{x_1, x_2, x_3, x_7\}$ ,

300  $(A_1 + B_1)^{\circ}X = \{x_1, x_2, x_3, x_7, x_8\}$ . And  $(A_2 + B_2)^{\circ}X = \{x_1, x_3, x_7\}$ ,  $(A_2 + B_2)^{\circ}X = \{x_1, x_2, x_3, x_4, x_7, x_8\}$ .

301 Then  $(N_1 + N_2)^{\circ}X = \{x_1, x_2, x_3, x_7\}$ ,  $(N_1 + N_2)^{\circ}X = \{x_1, x_2, x_3, x_4, x_7, x_8\}$ .

302 From Example 2, it follows that  $A_1 \cup B_1X = \{x_1, x_2, x_3, x_7\}$ ,  $\overline{A_1 \cup B_1}X = \{x_1, x_2, x_3, x_7\}$ .

303 For  $U/(A_2 \cup B_2) = \{\{x_1\}, \{x_2, x_8\}, \{x_3\}, \{x_4\}, \{x_6\}, \{x_7\}\}$ , then  $(A_2 \cup B_2)X = \{x_1, x_3, x_7\}$ ,  $(A_2 \cup B_2)X = \{x_1, x_2, x_3, x_7, x_8\}$ .

304 In addition,  $U/((A_1 \cup B_1) \cup (A_2 \cup B_2)) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}, \{x_8\}\}$ , one has  $(N_1 \cup N_2)X = (A_1 + B_1)^{\circ}X \cup$   
 305  $(A_2 + B_2)^{\circ}X = \{x_1, x_2, x_3, x_7\}$  and  $\overline{N_1 \cup N_2}X \sim (\overline{N_1 \cup N_2})(\sim X) = \{x_1, x_2, x_3, x_7\}$ .

306 Obviously, for the optimistic 2-type neighborhood multigranulation rough set, we have that  $(N_1 + N_2)^{\circ}X = \{x_3, x_7\} \subseteq$   
 307  $\{x_1, x_3, x_7\} = (N_1 \cup N_2)X$ , and  $(N_1 + N_2)^{\circ}X = \{x_1, x_2, x_3, x_7, x_8\} \supseteq \{x_1, x_2, x_3, x_7\} = \overline{(N_1 \cup N_2)}X$ .

308 From the definition of approximation and the discussion above, we can get the following properties of the lower and  
 309 upper approximations.

310 **Proposition 5.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2$  two neighborhood relations on the universe  
 311  $U$ . Then for any  $X \subseteq U$ , then

312 (1)  $(N_1 + N_2)^{\circ}X \subseteq (N_1 \cup N_2)X$ ,

313 (2)  $(N_1 + N_2)^{\circ}X \supseteq (N_1 \cup N_2)X$ .

314 **Proof.** (1) For any  $x \in (N_1 + N_2)^{\circ}X$ , from Definition 5, it follows that  $x \in n_{(A_1+B_1)}$  and  $x \in n_{(A_2+B_2)}$ . Hence,  $x \in n_{(A_1+B_1)}(x) \cap$   
 315  $n_{(A_2+B_2)}(x)$ ,  $n_{(A_1+B_1)}(x) \cap n_{(A_2+B_2)}(x) \subseteq n_{(N_1 \cup N_2)}(x)$ , we have  $x \in (N_1 \cup N_2)X$ , i.e.,  $(N_1 + N_2)^{\circ}X \subseteq (N_1 \cup N_2)X$ .

316 (2) Due to duality property of the lower and upper approximations,  $(N_1 \cup N_2)X \sim (\overline{N_1 \cup N_2})(\sim X)$ . Applying the  
 317 result of (1), we have that  $\overline{(N_1 \cup N_2)}X \sim (\overline{N_1 \cup N_2})(\sim X) \subseteq \sim (N_1 + N_2)^{\circ}(\sim X) = \overline{(N_1 + N_2)^{\circ}X}$ , i.e.,  $\overline{(N_1 \cup N_2)}X \subseteq$   
 318  $\overline{(N_1 + N_2)^{\circ}X}$ .

319 This completes the proof.  $\square$

320 **Corollary 6.**  $Bn_{N_1}(X) \cup Bn_{N_2}(X) \subseteq Bn_{(N_1+N_2)}^{\circ}(X)$ .

**Corollary 7.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2$  two neighborhood relations on the universe  
 $U$ . Then, for  $X \subseteq U$ , one has

$$\alpha_{(N_1+N_2)}^{\circ}(X) \leq \alpha_{(N_1 \cup N_2)}(X).$$

321 **Proof.** This is straightforward from the definition of the accuracy measure of  $X$ .  $\square$

322 **Proposition 6.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2$  two neighborhood relations on the universe  
 323  $U$ , and  $X \subseteq U$ . If  $\delta_1, \delta_2$  are two nonnegative numbers and  $\delta_1 \geq \delta_2$ , then

324 (1)  $(N_1 + N_2)_{\delta_1}^{\circ}X \subseteq (N_1 + N_2)_{\delta_2}^{\circ}X$ ,

325 (2)  $(N_1 + N_2)_{\delta_1}^{\circ}X \supseteq (N_1 + N_2)_{\delta_2}^{\circ}X$ .

326 **Proof.** It can be easily proved similar to Proposition 3.

327 Proposition 6 states that the size of lower approximation of  $X$  under a 2-type optimistic neighborhood-based multigran-  
 328 uation rough set will become much larger with the value of the parameter  $\delta$  being much bigger. Its upper approximation  
 329 has the inverse conclusion.  $\square$

**Corollary 8.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2$  two neighborhood relations on the universe  $U$ , and  $X \subseteq U$ . If  $\delta_1, \delta_2$  are two nonnegative numbers and  $\delta_1 \geq \delta_2$ , then,

$$\alpha_{(N_1+N_2)\delta_1}^0(X) \leq \alpha_{(N_1+N_2)\delta_2}^0(X).$$

**Proposition 7.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2$  two neighborhood relations on the universe  $U$ , and  $X, Y \subseteq U$ . If  $X \subseteq Y$ , then

332 (1)  $\underline{(N_1 + N_2)}^0 X \subseteq \underline{(N_1 + N_2)}^0 Y$ ,

333 (2)  $\overline{(N_1 + N_2)}^0 X \subseteq \overline{(N_1 + N_2)}^0 Y$ .

334 **Proof.** (1) If  $X \subseteq Y, X \cap Y = X$ . Then

335  $\underline{(N_1 + N_2)}^0 X = \underline{(N_1 + N_2)}^0 (X \cap Y)$

336  $= \underline{N_1(X \cap Y)} \cup \underline{N_2(X \cap Y)}$

337  $= \underline{(N_1 X \cap N_1 Y)} \cup \underline{(N_2 X \cap N_2 Y)}$

338  $= \underline{((N_1 X \cap N_1 Y) \cup N_2 X) \cap ((N_1 X \cap N_1 Y) \cup N_2 Y)}$

339  $= \underline{(N_1 X \cup N_2 X) \cap (N_1 Y \cup N_2 Y) \cap (N_1 X \cup N_2 Y) \cap (N_1 Y \cup N_2 X)}$

340  $= \underline{(N_1 + N_2)}^0 X \cap \underline{(N_1 + N_2)}^0 Y \cap \underline{(N_1 Y \cup N_2 X) \cap (N_1 X \cup N_2 Y)}$

341  $\subseteq \underline{(N_1 + N_2)}^0 X \cap \underline{(N_1 + N_2)}^0 Y$

342  $\subseteq \underline{(N_1 + N_2)}^0 Y$ .

343 So  $\underline{(N_1 + N_2)}^0 X \subseteq \underline{(N_1 + N_2)}^0 Y$ .

344 (2) If  $X \subseteq Y, \sim X \supseteq \sim Y$ , from the result of (1),  $\underline{(N_1 + N_2)}^0(\sim X) \supseteq \underline{(N_1 + N_2)}^0(\sim Y)$ . Then,  $\sim \underline{(N_1 + N_2)}^0$

345  $(\sim X) \subseteq \sim \underline{(N_1 + N_2)}^0(\sim Y)$ , then  $\overline{(N_1 + N_2)}^0 X \subseteq \overline{(N_1 + N_2)}^0 Y$ .

346 This completes the proof.  $\square$

Similarly, the pessimistic 2-type neighborhood multigranulation rough set with two neighborhood granulations can be also defined as follows:

$$\underline{(N_1 + N_2)}^P X = \{x \mid n_{(A_1+B_1)}(x) \subseteq X \wedge n_{(A_2+B_2)}(x) \subseteq X\}, \tag{13}$$

$$\overline{(N_1 + N_2)}^P X = \sim \underline{(N_1 + N_2)}^P(\sim X). \tag{14}$$

The area of uncertainty or boundary region is defined as:

$$Bn_{(N_1+N_2)}^P(X) = \overline{(N_1 + N_2)}^P X \setminus \underline{(N_1 + N_2)}^P X.$$

347 Parallely, we can present the corresponding properties of this pessimistic version.

348 Based on the above conclusions, we extend 2-type NMGRS based on two neighborhood relations to that based on multiple  
349 neighborhood relations.

**Definition 6.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $A_1, A_2, \dots, A_m$  categorical attribute subsets of  $AT$ ;  $B_1, B_2, \dots, B_m$  numerical attributes of  $AT$ ,  $N_i$  induced by  $A_i$  and  $B_i$  for  $i = 1, 2, \dots, m$ , and  $X \subseteq U$ . We define an optimistic multigranulation lower approximation and an upper approximation of  $X$  by the following:

$$\sum_{i=1}^m \underline{N_i}^0 X = \bigcup \{x \in U \mid n_{(A_i+B_i)}(x) \subseteq X, i \leq m\}, \tag{15}$$

$$\overline{\sum_{i=1}^m N_i}^0 X = \sim \sum_{i=1}^m \underline{N_i}(\sim X). \tag{16}$$

Similarly, the area of uncertainty or boundary region is defined as:

$$Bn_{\sum_{i=1}^m N_i}^0(X) = \overline{\sum_{i=1}^m N_i}^0 X \setminus \underline{\sum_{i=1}^m N_i}^0 X.$$

350 We call  $(\underline{\sum_{i=1}^m N_i}^0 X, \overline{\sum_{i=1}^m N_i}^0 X)$  an optimistic 2-type NMGRS based on multiple neighborhood relations.

351 **Proposition 8.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2, \dots, N_m$   $m$  neighborhood relations on the  
352 universe  $U$ , and  $X \subseteq U$ . Then,

353 (1)  $\sum_{i=1}^m N_i^0 X \subseteq (N_1 \cup N_2 \cup \dots \cup N_m)X$ ,

354 (2)  $\sum_{i=1}^m N_i^0 X \supseteq \overline{(N_1 \cup N_2 \cup \dots \cup N_m)X}$ .

355 **Proof.** If  $m = 1$ , they are straightforward.

356 If  $m > 1$ , we prove them as follows:

357 (1) It can be easily proved from Definition 6.

358 (2)  $\sum_{i=1}^m N_i^0 X = \sim \sum_{i=1}^m N_i^0 (\sim X) \supseteq \sim (N_1 \cup N_2 \cup \dots \cup N_m)(\sim X) = \overline{(N_1 \cup N_2 \cup \dots \cup N_m)X}$ .

359 This completes the proof.  $\square$

**Corollary 9.** Let  $NIS = (U, AT, N)$  be a neighborhood system,  $N_1, N_2, \dots, N_m$   $m$  neighborhood relations on the universe  $U$ , and  $X \subseteq U$ . Then,

360  $\alpha_{\sum_{i=1}^m N_i}^0(X) \leq \alpha_{(N_1 \cup N_2 \cup \dots \cup N_m)}(X)$ .

361 **Proposition 9.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2, \dots, N_m$   $m$  neighborhood relations on the  
362 universe  $U$ ,  $X \subseteq U$ , and  $\delta_1, \delta_2$  two nonnegative numbers. If  $\delta_1 \geq \delta_2$ , then,

363 (1)  $(\sum_{i=1}^m N_i)_{\delta_1}^0 X \subseteq (\sum_{i=1}^m N_i)_{\delta_2}^0 X$ ,

364 (2)  $(\sum_{i=1}^m N_i)_{\delta_1}^0 X \supseteq (\sum_{i=1}^m N_i)_{\delta_2}^0 X$ .

365 **Proof.** It can be proved similar to Proposition 3.  $\square$

**Corollary 10.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2, \dots, N_m$   $m$  neighborhood relations on the  
universe  $U$ , and  $X \subseteq U$ . If  $\delta_1, \delta_2$  are two nonnegative numbers, and  $\delta_1 \geq \delta_2$ , then the following properties hold.

$\alpha_{(\sum_{i=1}^m N_i)_{\delta_1}}^0(X) \leq \alpha_{(\sum_{i=1}^m N_i)_{\delta_2}}^0(X)$ .

366 **Proposition 10.** Let  $NIS = (U, AT, N)$  be a neighborhood information system,  $N_1, N_2, \dots, N_m$   $m$  neighborhood relations on the  
367 universe  $U$ , and  $X, Y \subseteq U$ . If  $X \subseteq Y$ , then

368 (1)  $\sum_{i=1}^m N_i^0 X \subseteq \sum_{i=1}^m N_i^0 Y$ ,

369 (2)  $\sum_{i=1}^m N_i^0 X \subseteq \sum_{i=1}^m N_i^0 Y$ .

370 **Proof.** It is similar to the proof of Proposition 4.  $\square$

Similarly, we can also define the pessimistic 2-type neighborhood multigranulation rough set as the following:

$$\sum_{i=1}^m N_i^P X = \{x \in U \mid n_{(A_1+B_1)}(x) \subseteq X \wedge \dots \wedge n_{(A_m+B_m)}(x) \subseteq X\}, \tag{17}$$

$$\overline{\sum_{i=1}^m N_i^P X} = \sim \sum_{i=1}^m N_i^P (\sim X). \tag{18}$$

Similarly, the area of uncertainty or boundary region is defined as:

$$Bn_{\sum_{i=1}^m N_i}^P(X) = \overline{\sum_{i=1}^m N_i^P X} \setminus \sum_{i=1}^m N_i^P X.$$

371

372 Analogously, we can gain the same results of the pessimistic version with multiple neighborhood granulations.

373 **4. Attribute reduction of neighborhood multigranulation rough sets**

374 In this section, we investigate the reduction of coverings induced by the multiple neighborhood relations. A discernibility  
 375 matrix will be used to compute all the reducts of neighborhood multigranulation rough set. The objective of reduction is to  
 376 select a subset of coverings that can preserve consistence of the neighborhood decision system [1]. Let  $\Omega = \{C_1, C_2, \dots, C_m\}$   
 377 be a family of coverings of  $U$ .  $C_i = \{K_{i1}, K_{i2}, \dots, K_{it_i}\}$ , where  $K_{ij}$  is nonempty subset of  $U$  for  $j = \{1, 2, \dots, t_i\}$ . For any  $x \in U$ ,  
 378  $(C_i)_x = \bigcap \{K_{ij} \mid K_{ij} \in C_i, x \in K_{ij}\}$ ,  $Cov(C_i) = \{(C_i)_x \mid x \in U\}$ ,  $\Omega_x = \bigcap \{K_{ix} \in Cov(C_i), x \in C_{ix}\}$ , and  $Cov(\Omega) = \{\Omega_x \mid x \in U\}$ .  
 379 As a result,  $Cov(C_i) = \{(C_i)_x \mid x \in U\}$  and  $Cov(\Omega) = \{\Omega_x \mid x \in U\}$  are two coverings of  $U$ .

380 **Definition 7.** Let  $\Omega = \{C_1, C_2, \dots, C_m\}$  be a family of coverings of  $U$ ,  $D = \{d\}$  a decision attribute set, and  $U/D =$   
 381  $\{D_1, D_2, \dots, D_q\}$  a decision partition on  $U$ . If for any  $x \in U$ , there exists  $D_j \in U/D$  such that  $\Omega_x \subseteq D_j$ , then decision system  
 382  $(U, \Omega, D)$  is called a consistent covering decision system and denoted by  $Cov(\Omega) \leq U/D$ .

383 **Definition 8.** Let  $NIS = (U, AT \cup D, N)$  be a neighborhood decision information system, where  $D = \{d\}$ ,  $C_i$  induced by a  
 384 categorical attribute subset  $A_i$  or a numerical attribute subset  $B_i$ ,  $i = 1, 2, \dots, m$ , and  $\Omega = \{C_1, C_2, \dots, C_m\}$   $m$  coverings of  
 385  $U$ . We call  $(U, \Omega, D)$  a covering neighborhood decision system.

386 **Definition 9.** Let  $(U, \Omega, D = \{d\})$  be a covering neighborhood decision information system. For  $C_i \in \Omega$ , if  $Cov(\Omega - C_i) \leq$   
 387  $U/D$ , then  $C_i$  is called a superfluous covering relative to  $D$  in  $\Omega$ , otherwise  $C_i$  is called indispensable relative to  $D$  in  $\Omega$ . For every  
 388  $P \subseteq \Omega$  satisfying  $Cov(P) \leq U/D$ , if every element in  $P$  is an indispensable covering, i.e., for any  $C_i \in P$ , if  $Cov(P - C_i) \not\leq U/D$ ,  
 389 then  $P$  is called a relative reduct of  $\Omega$  relative to  $D$ . The disjunction of all the indispensable elements in  $\Omega$  is called the core  
 390 of  $\Omega$  to  $D$ , denoted by  $NCore_D(\Omega)$ . The relative reduct of a consistent covering decision system is the subset of coverings to  
 391 ensure the consistency of the decision information system.

392 When the attribute reduction of a neighborhood-based multigranulation rough set is to calculate, we will employ the  
 393 discernibility matrix approach proposed by Chen et al. for this objective, which is as follows:

**Definition 10 [1].** Let  $(U, \Omega, D = \{d\})$  be a consistent covering decision system. Suppose  $U = \{x_1, x_2, \dots, x_n\}$ , by  
 $M(U, \Omega, D)$ , we denote a  $n \times n$  matrix  $(c_{ij})$ , called the discernibility matrix of  $(U, \Omega, D = \{d\})$ , defined as

$$c_{ij} = \begin{cases} \{C \in \Omega : (C_{x_i} \not\subseteq C_{x_j}) \wedge (C_{x_j} \not\subseteq C_{x_i})\} \cup \{C_s \wedge C_t : (C_{sx_i} \subset C_{x_j}) \wedge (C_{sx_j} \subset C_{x_i})\}, & d(\Omega_{x_i}) \neq d(\Omega_{x_j}), \\ \Omega, & d(\Omega_{x_i}) = d(\Omega_{x_j}). \end{cases}$$

394 In which  $D = \{d\}$  and  $d(x)$  is a decision function  $d : U \rightarrow V_d$  of the universe  $U$  into value set  $V_d$ . For every  $x_i, x_j \in U$ , if  
 395  $\Omega_{x_i} \subseteq \Omega_{x_j}$ , then  $d(x_i) = d([x_i]_D) = d(\Omega_{x_i}) = d(\Omega_{x_j}) = d(x_j) = d([x_j]_D)$ . If  $d(\Omega_{x_i}) \neq d(\Omega_{x_j})$ , then  $\Omega_{x_i} \cap \Omega_{x_j} = \emptyset$ , i.e.,  
 396  $\Omega_{x_i} \not\subseteq \Omega_{x_j}$  and  $\Omega_{x_j} \not\subseteq \Omega_{x_i}$ . But if  $\Omega_{x_i} \not\subseteq \Omega_{x_j}$  and  $\Omega_{x_j} \not\subseteq \Omega_{x_i}$ , then either  $d(\Omega_{x_i}) = d(\Omega_{x_j})$  or  $d(\Omega_{x_i}) \neq d(\Omega_{x_j})$  are possible. For  
 397 this case, if  $\Omega_{x_i} \cap \Omega_{x_j} \neq \emptyset$ , we have  $d(\Omega_{x_i}) = d(\Omega_{x_j})$ . If  $d(\Omega_{x_i}) = d(\Omega_{x_j})$ , then both  $\Omega_{x_i} \not\subseteq \Omega_{x_j}$  and  $\Omega_{x_j} \not\subseteq \Omega_{x_i}$ , or  $\Omega_{x_i} \subseteq \Omega_{x_j}$   
 398 or  $\Omega_{x_j} \subseteq \Omega_{x_i}$  are possible.

399 In the following, we give an example to illustrate the covering reduct of 1-type neighborhood multigranulation rough  
 400 set through using the discernibility matrix approach proposed by Chen et al. The covering reduct of 2-type neighborhood  
 401 multigranulation rough set can be done similarly.

402 **Example 4.** Table 2 depicts a neighborhood decision information system  $NIS = (U, AT \cup \{d\}, N)$  in which  $AT = \{\text{outlook},$   
 403  $\text{temperature}, \text{windy}\}$ ,  $\{d\} = \{\text{play}\}$ . The numerical attribute value of *temperature* is standardized into  $[0, 1]$  (see [6]) for  
 404 computing and we suppose  $\delta = 0.1$ . By Definition 2, we have that:

- 405 Let  $P_1 = \{O\}$ , then  $C_1 = \{\{x_1, x_2, x_8\}, \{x_3, x_7\}, \{x_4, x_5, x_6\}\}$ .
- 406 Let  $P_2 = \{T\}$ , then  $C_2 = \{\{x_1, x_2, x_3\}, \{x_2, x_1, x_3, x_4, x_8\}, \{x_3, x_1, x_2\}\}, \{x_4, x_2, x_5, x_6, x_7, x_8\}; \{x_5, x_4, x_6, x_7, x_8\},$   
 407  $\{x_6, x_4, x_5, x_7, x_8\}, \{x_7, x_4, x_5, x_6, x_8\}, \{x_8, x_2, x_4, x_5, x_6, x_7\}\}$ .
- 408 Let  $P_3 = \{W\}$ , then  $C_3 = \{\{x_1, x_3, x_4, x_5, x_8\}, \{x_2, x_7, x_6\}\}$ .
- 409 Let  $P_4 = \{O, T\}$ , then  $C_4 = \{\{x_1, x_2\}, \{x_2, x_1, x_8\}, \{x_3\}, \{x_4, x_5, x_6\}, \{x_7\}, \{x_8, x_2\}\}$ .

**Table 2**  
A playing tennis information system with mixed attributes.

	Outlook	Temperature	Windy	Play
$x_1$	Sunny	85	False	No
$x_2$	Sunny	80	True	No
$x_3$	Overcast	83	False	Yes
$x_4$	Rainy	70	False	Yes
$x_5$	Rainy	68	False	Yes
$x_6$	Rainy	65	True	No
$x_7$	Overcast	64	True	Yes
$x_8$	Sunny	72	False	No





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