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Determining decision makers' weights in group ranking: a granular computing method

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Abstract Deriving the consensus ranking(s) from a set of rankings plays an important role in group decision making. However, the relative importance, i.e. weight of a decision maker, is ignored in most of the ordinal ranking methods. This paper aims to determine the weights of decision makers by measuring the support degree of each pair of ordinal rankings. We first define the similarity degree of dominance granular structures to depict the mutual relations of the ordinal rankings. Then, the support degree, which is obtained from similarity degree, is presented to determine weights of decision makers. Finally, an improved programming model is proposed to compute the consensus rankings by minimizing the violation with the weighted ranking(s). Two examples are given to illustrate the rationality of the proposed model.

Keywords Total ranking · Partial ranking · Similarity degree · Support degree · Granular computing

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1 Introduction

Group decision-making (GDM) is accentuated in the organizations when they realize that the complexity and the size of their decision problems could not be solved by working individually [32]. In practical GDM problems, decision makers (DMs) provide their preference information in multiple formats, such as utility functions [15, 33, 39, 43], multiplicative preference relations [37, 38], fuzzy preference relations [4, 16, 24], and ordinal rankings [1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 18, 19, 20, 21, 22]. Many real world decision problems involve ordinal rankings, such as marketing of new products [9], allocating priorities to R & D projects [13], service evaluation of e-commerce [23], document retrieval [46]. The attractiveness of the ordinal ranking format is due in great part to the minimal amount of information required, i.e. each decision maker only needs to express a preference of one object over another, not the degree of preference as it is required in utility functions and preference relations.

Aggregating the consensus ranking(s) from a set of individual ordinal rankings has been investigated by several researchers. There are mainly two approaches in the consensus ranking problems: total ranking and partial ranking. The total ranking approach demands that decision makers appraise all the objects and deliver a total rank of the objects. The total ranking approach has been studied more maturely. The well-known Borda-Kendall [26] method is the representative total ranking approach. Kemeny and Snell [25] first proposed the optimal approach that minimizes the distance between the solution ranking and the set of given rankings in pairwise format. Cook designed another optimal model to minimize the number of disagreements among the given total ranking [8, 9].

In many practical problems, it might be better to allow decision makers to provide partial rankings on the object set. For instance, a film review website wants to launch a most popular movies recommender project and the workers make a survey of one thousand college students on naming several Hollywood Movies of 2012 that they had seen and ranking them in decreasing order of preference. Then student A might give the preference as: *Ice Age: Continental Drift*, *Madagascar3*, *Skyfall*, *The Vow*, *Dredd*, and other movies are not compared. Student B might give his preference as: *Skyfall*, *Looper*, *Argo*, and some other movies are not compared. Both of A's list and B's list are partial rankings of movie titles of 2012. Another instance is the peer review of articles in the academic community. In most cases, the committee member can only express the preferences concerning a proper subset of the articles for the the domain expertise or the limited time and attentions. For the reason that the decision makers have no preference knowledge about the two objects, we can think that no comparison between two objects means the incomparable relation for the current judgement of each decision maker. Several researchers have paid their attention to this kind of problems. Bogart [1, 2] analyzed the structure of transitive and asymmetric preference relations, defined the distance metric of the two kinds of relations, and concluded that the mean of a collection of relations is the consensus preference. Cook et al. studied the preference in priority ranking [7], compared the relations of the different methods [11, 12] and constructed the strict mathematical programming model to solve the partial ordinal ranking problems [12, 13]. Brüggemann [3] used a local partial order model to estimate the averaged ranks. Cook et al. [13] proposed a

Branch-and-Bound approach to support the construction aggregate ranking(s) so as to achieve the maximum possible overlap among the partial rankings provided by different decision makers. The Branch-and-Bound method given in [13] is one of the most effective methods for solving the partial ranking problems by discarding large subsets of fruitless rankings through computing the upper and lower estimated bounds of the branched nodes.

Weights of decision makers play a very important role in GDM problems. It is an interesting research topic to determine weights of decision makers. In many group decision making literature, particularly in ordinal ranking problems [1,2,7,8,9,10, 11,12,13], relative importance, i.e. weight, has been largely ignored. It is typically assumed that all decision makers in a group have the same importance. Some researchers have studied the weight determination methods in ordinal ranking problems. The subjective and objective methods are two kinds of weights determining approaches. Ramanathan [36] used the eigenvector method, a subjective method, to determine the weights of group members. Emond and Mason [14] determined the objective weights of decision makers by computing the consensus degree of each decision maker. Jabeur et al. [18,19,20,21,22] used a subjective and objective combined method to obtain the relative importance coefficient of each decision maker under each pair of objects.

Although the above approaches have done beneficial attempts to derive the consensus ranking(s) by considering the weights of decision makers, the relations among the total or partial preference rankings are greatly ignored and these information indeed provides very useful clues for determining weights of decision makers. Granular computing, which is seen as a conceptual and algorithmic platform supporting analysis and design of human-centric intelligent systems, has attracted the attention of many researchers [27,34,41,42,43,44,47]. Granule, granulation and granularity are regarded as the three primitive notions of GrC. A granule is a clump of objects drawn together by indistinguishability, similarity and proximity of functionality. Granulation of an object leads to a collection of granules. The granularity is the measurement of the granulation degree of objects [35]. For the less comparison information, we can not use the entropy or deviation measures [5,40], which are widely used in the utility expressed decision making, to calculate the weights, and the clues that we can use are structures of the rankings. Granularity measure is indeed a good tool to reveal the relations among different granular structures. In this paper, we try to design a new granularity measure in the scheme of granular computing to analyze the support degrees among different decision makers. In real life, if one's opinion is approved by most of the decision makers, he or she is usually considered an influencer and should be granted a comparatively larger power. Firstly, we construct the dominance granular structures by granulating the dominating objects under the corresponding ordinal rankings. Inspired by the measurements of knowledge granularities in the studies of Refs. [17,28,30,29,34,48], we define the similarity degree of each pair of dominance granular structures to analyze the differences among decision makers' judgments. Then, an improved similarity degree, support degree, is used to determine the weights of decision makers in ordinal group ranking. In addition, we improve the programming model to derive the consensus ranking(s) of the weighted ordinal rankings by the Branch-and-Bound method proposed in [13].

The rest of this paper is organized as follows. The ordinal ranking and its dominance granular structure are presented in Subsection 2.1; the mathematical programming model for deriving consensus total ranking(s) with minimum violations is followed in Subsection 2.2. Section 3 defines the similarity degree and support degree of dominance granular structures to determine weights of decision makers. Considering the weights of decision makers, Section 4 improves Cook's minimum violation model to compute the consensus ranking(s) from a set of weighted ordinal rankings. Two examples are given in Section 5 to illustrate the rationality of the proposed method. Section 6 concludes the paper.

2 Preliminaries

In this section, several notions related to rankings are given in the first subsection. Then the mathematical programming model for deriving the consensus ranking(s) from an ordinal ranking set are reviewed in the second subsection.

2.1 Total ranking, partial ranking and ranking

Definition 1 [31] Let $X=\{x_1,x_2,\ldots,x_n\}$ $(n>1)$ and R be a binary relation on X. If R satisfies the following conditions:

(1) for any $x\in X$, $(x,x)\in R$ (reflexivity);

(2) if $(x,y)\in R$ and $(y,x)\in R$, then $x=y$ (antisymmetric);

(3) if $(x,y)\in R$ and $(y,z)\in R$, then $(x,z)\in R$ (transitivity);

then, R is called a partial relation on X.

$(x,y)\in R$ is simplified as $x\succeq_R y$. In decision making, it is senseless to compare one object to itself, so we mainly investigate the irreflexive partial relation, in which x prefers y is denoted as $x\succ_R y$.

Definition 2 Let $X=\{x_1,x_2,\ldots,x_n\}$ $(n>1)$ and R be a strict partial relation on X. The set $VES(R)=\{x\mid x,y\in X,\exists y,x\succ_R y\ or\ y\succ_R x\}$ is called the valid evaluation set of R.

Let R be a strict partial relation on X. If $x\in VES(R)$, then x is called a compared object; if $x\in X-VES(R)$, then x is not compared with any other objects.

Definition 3 [31] Let $X=\{x_1,x_2,\ldots,x_n\}$ $(n>1)$ and R be a strict partial relation on X. If for any two distinct elements $x,y\in X$, $x\succ_R y$ or $y\succ_R x$, then R is called **total** (complete) **ranking** on X.

Suppose R is a total ranking on X, then the ranking can be clearly expressed as $x_{i_1}\succ_R x_{i_2}\succ_R\ldots\succ_R x_{i_n}$. Let $\mathbb{R}^X$ represent the set of all the total rankings on X, then $|\mathbb{R}^X|=n!$.

Example 1 Let $X=\{x_1,x_2,x_3,x_4\}$, $R_1=\{(x_2,x_1)\}$, $R_2=\{(x_2,x_1),(x_2,x_3),(x_2,x_4),(x_3,x_1),(x_3,x_4),(x_4,x_1)\}$. It is easy to verify that R_1 and R_2 are two strict partial relations on X. $VES(R_1)=\{x_1,x_2\}$, $VES(R_2)=\{x_1,x_2,x_3,x_4\}$. In addition, R_2 is a total ranking and it can be denoted as $x_2\succ_{R_2}x_3\succ_{R_2}x_4\succ_{R_2}x_1$.

Definition 4 Let $X = \{x_1, x_2, \ldots, x_n\}$ $(n > 1)$, R be a strict partial relation on X and $Y \subseteq X$. $Proj_Y(R) = (Y \times Y) \cap R$ is called the projection of R on Y.

Property 1 Let $X = \{x_1, x_2, \ldots, x_n\}$ $(n > 1)$, R be a strict partial relation on X and $Y \subseteq X$. $Proj_Y(R)$ is a strict partial relation on Y.

Proof: (1) "Irreflexive." For any $y \in Y$, since $Y \subseteq X$, $y \in X$. R is irreflexive on X, so $(y,y) \notin R$, therefore, $(y,y) \notin R \cap (Y \times Y) = Proj_Y(R)$.

(2) "Antisymmetric." If $(x,y) \in Proj_Y(R)$ and $(y,x) \in Proj_Y(R)$, then $(x,y) \in R$ and $(y,x) \in R$. Since R is antisymmetric, it follows $y = x$.

(3) "Transitivity." Similar to the proof of (2).

This completes the proof.

Property 1 shows that the projection of a strict partial relation is also a strict partial relation.

Definition 5 Let $X = \{x_1, x_2, \ldots, x_n\}$ $(n > 1)$ and R be a strict partial relation on X. If $VER(R) \neq X$ and $Proj_{VER(R)}(R)$ is a total ranking on $VER(R)$, then we call R a **partial ranking** on X.

A partial ranking on X can be seen a total ranking on a proper subset of X. The object set X are divided into two non-empty parts: $VER(R)$ and $X - VER(R)$. If both of x and y are in $VER(R)$, then there exists an explicit order relation between x and y; otherwise, the two elements x and y are incomparable.

Remark 1 From Definition 5, a total ranking is different from a partial ranking. All the objects are comparable in a total ranking. In contrast, some of the objects are incomparable in a partial ranking. Total ranking and partial ranking are collectively called **ranking**.

Definition 6 Let $X = \{x_1, x_2, \ldots, x_n\}$ $(n > 1)$ and R be a ranking on X. For any $x \in X$, the weak dominance granule of x with respect to the ranking R is defined as

$$[x]_R^{\succ} = \{y \mid y \succ_R x \textit{ and } y \in X\} \cup \{x\}. \tag{1}$$

The weak dominance granule $[x]_R^{\succ}$ is the set that includes x and the elements dominate x with respect to R. The elements except x in a weak dominance granule, $[x]_R^{\succ}$, are drawn together by the preference relation.

Let $X/R = \{[x]_R^{\succ} \mid x \in X\}$. X/R is called a dominance granular structure of X induced by R. Each ranking induces an unique dominance granular structure. The weak dominance granules in X/R do not constitute a partition of X, in general, they constitute a covering of X. In the following section, we analyze the relations among decision makers by comparing the differences among the dominance granular structures of rankings provided by the corresponding decision makers.

Example 2 (Continued from Example 1.) We calculate the dominance granule structures with respect to R_1 and R_2 given in Example 1.

$$X/R_1 = \{[x_1]_{R_1}^{\succ}, [x_2]_{R_1}^{\succ}, [x_3]_{R_1}^{\succ}, [x_4]_{R_1}^{\succ}\},$$

where $[x_1]_{R_1}^{\succ} = \{x_1, x_2\}$, $[x_2]_{R_1}^{\succ} = \{x_2\}$, $[x_3]_{R_1}^{\succ} = \{x_3\}$, $[x_4]_{R_1}^{\succ} = \{x_4\}$.

$$X/R_2 = \{[x_1]_{R_2}^{\succ}, [x_2]_{R_2}^{\succ}, [x_3]_{R_2}^{\succ}, [x_4]_{R_2}^{\succ}\},$$

where $[x_1]_{R_2}^{\succ} = \{x_1, x_2, x_3, x_4\}$, $[x_2]_{R_2}^{\succ} = \{x_2\}$, $[x_3]_{R_2}^{\succ} = \{x_2, x_3\}$, $[x_4]_{R_1}^{\succ} = \{x_2, x_3, x_4\}$.

2.2 Mathematical programming model for deriving the consensus ranking(s) with minimum violations

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of objects, $DM = \{e_1, e_2, \ldots, e_l\}$ be a set of decision makers and $\mathfrak{R} = \{\succ_{R_1}, \succ_{R_2}, \ldots, \succ_{R_l}\}$ be a set of rankings on X provided by the decision makers in DM. The task of us is to find the total ranking(s) on X with minimum violations of the rankings in $\mathfrak{R}$.

Cook et al. [12] solved the pairwise comparison consensus ranking problem from a strictly mathematical programming perspective. The minimum violation consensus total ranking problem is expressed as an integer programming formulation as follows.

$$\begin{aligned} \min_{R\in\mathbb{R}^X} \; & M(R) = \sum_{(x_i,x_j)\in X^2: i\neq j} p_{ij} v_{ij} \\ s.t. \; & p_{ij} + p_{jk} \leq 1 + p_{ik} \quad \forall (x_i, x_j, x_k) \in X^3 : i \neq j,\ i \neq k, k \neq j, \\ & p_{ij} + p_{ji} = 1 \quad \forall (x_i, x_j) \in X^2, i \neq j, \\ & p_{ij} \in \{0, 1\}. \end{aligned} \tag{2}$$

where $v_{ij} = |\{R_k | x_j \succ_{R_k} x_i, k = 1, 2, \ldots, l\}|$ represents the number of violations occurs in $\mathfrak{R}$ if x_i is ranked ahead of x_j in the final ranking R; if $x_i \succ_R x_j$ $p_{ij} = 1$, otherwise, $p_{ij} = 0$.

In the above programming formulation, the evaluation information of decision makers is expressed via the violation matrix $V = (v_{ij})_{n\times n}$. The solutions of the programming problem are total rankings. It is deserved to point out that the consensus ranking(s) can be derived theoretically by solving the above problem, however, size becomes a major issue for the reason that the number of constraints is given by $n(n-1)(n-2) + \frac{n(n-1)}{2}$. Cook et al. have proved that Eq. (2) can be re-expressed as the unconstrained programming problem.

$$\min_{R\in\mathbb{R}^X} \quad \mathfrak{M}(R) = \sum_{x_i \succ_R x_j} v_{ij}. \tag{3}$$

An effective Branch-and-Bound algorithm was given in [12] to solve the unconstrained programming model. They assumed that all members have the same importance. In fact, the differences among the decision makers's skills, experience and personality imply that they should be granted with different weights in the overall decision making processes. In the following sections, we use a granular computing method to determine the weights of decision makers and derive the consensus total ranking(s) from a weighted strict ordinal ranking set.

All of the symbols utilized in this paper are summarized in Table 1.

3 Support degree based method for determining the weights of decision makers

Different decision makers may deliver their opinions by different rankings (including total ranking and partial ranking) in ordinal decision making problems. In this section, we depict the weights of decision makers by defining the support degree of rankings. First, we measure the similarity degree of each pair of rankings.

Table 1 Summary of the symbols utilized in this paper

Symbol	Meaning	Symbol	Meaning
X	The set of objects.	x_i	The i^{th} object.
e_i	The i^{th} decision maker.	DM	The set of decision makers.
R	The partial relation.	$VES(R)$	The valid evaluation set of R.
$x_i \succ x_j$	x_i is preferred to x_j.	$\mathbb{R}^X$	The set of all total rankings on X.
$Proj_Y(R)$	The projection of R on X.	X/R	The dominance granular structure of X induced by R.
$\mathfrak{R}$	The set of rankings on X.	$sim_X(R_1,R_2)$	The similarity degree between R_1 and R_2.
V,V'	The violation matrixes.	$sup(R_1,R_2)$	The support degree of R_2 from R_1.
T_i	The support of e_i from others.	η_i	The weight of e_i.
n	The number of objects.	l	The number of decision makers.

Definition 7 Let $X = \{x_1,x_2,\cdots,x_n\}$ $(n > 1)$. R_1, R_2 are two rankings on X, and $X/R_1 = \{[x_i]^{\succ}_{R_1} \mid x_i \in X\}$, $X/R_2 = \{[x_i]^{\succ}_{R_2} \mid x_i \in X\}$. If $R_1 \neq \emptyset$ or $R_2 \neq \emptyset$, then the similarity degree between R_1 and R_2 is defined as

$$sim_X(R_1,R_2) = \frac{\sum_{i=1}^{n} \log_2 \frac{n}{|[x_i]^{\succ}_{R_1} \cup [x_i]^{\succ}_{R_2}|}}{\sum_{i=1}^{n} \log_2 \frac{n}{|[x_i]^{\succ}_{R_1} \cap [x_i]^{\succ}_{R_2}|}}. \quad (4)$$

For the convenience of analysis below, if $R_1 = R_2 = \emptyset$, the similarity degree between them is defined as 0. If there is only one universe X involved, $sim_X(R_1,R_2)$ can be simplified as $sim(R_1,R_2)$.

The similarity degree between two rankings are defined based on the calculations of sets that are the granules from the two corresponding dominance granular structures.

Example 3 Let $X = \{x_1,x_2,x_3,x_4,x_5\}$. $R_1 : x_1 \succ x_2 \succ x_4 \succ x_3$ and $R_2 : x_2 \succ x_1 \succ x_3 \succ x_5$ are two strict partial rankings on X. We use Equation (4) to compute $sim(R_1,R_2)$. $X/R_1 = \{\{x_1\},\{x_1,x_2\},\{x_1,x_2,x_3,x_4\},\{x_1,x_2,x_4\},\{x_5\}\}$. $X/R_2 = \{\{x_1,x_2\},\{x_2\}, \{x_1,x_2,x_3\},\{x_4\},\{x_1,x_2,x_3,x_5\}\}$. Then,

$$sim(R_1,R_2) = \frac{log_2\frac{5}{2} + log_2\frac{5}{2} + log_2\frac{5}{4} + log_2\frac{5}{3} + log_2\frac{5}{4}}{log_2\frac{5}{1} + log_2\frac{5}{1} + log_2\frac{5}{3} + log_2\frac{5}{1} + log_2\frac{5}{1}} = 0.4015.$$

In what follows, we discuss the properties of the similarity degree defined above.

Property 2 Let $X = \{x_1,x_2,\cdots,x_n\}$ $(n > 1)$. If R_1 and R_2 are two rankings on X, then

$$0 \leq sim(R_1,R_2) \leq 1.$$

Proof: Suppose $R_1 = R_2 = \emptyset$. According to the special case of Definition 7, $sim(R_1,R_2) = 0$.

Suppose $R_1 \neq \emptyset$. For any $x_i \in X$,

$$\{x_i\} \subseteq \left([x_i]^{\succ}_{R_1} \cap [x_i]^{\succ}_{R_2}\right) \subseteq \left([x_i]^{\succ}_{R_1} \cup [x_i]^{\succ}_{R_2}\right) \subseteq X,$$

then
$$1 \leq |[x_i]^{\succ}_{R_1} \cap [x_i]^{\succ}_{R_2}| \leq |[x_i]^{\succ}_{R_1} \cup [x_i]^{\succ}_{R_2}| \leq n.$$
So
$$1 \leq \frac{n}{|[x_i]^{\succ}_{R_1} \cup [x_i]^{\succ}_{R_2}|} \leq \frac{n}{|[x_i]^{\succ}_{R_1} \cap [x_i]^{\succ}_{R_2}|} \leq n$$
and
$$0 \leq \frac{\sum\limits_{i=1}^{n} \log_2 \frac{n}{|[x_i]^{\succ}_{R_1} \cup [x_i]^{\succ}_{R_2}|}}{\sum\limits_{i=1}^{n} \log_2 \frac{n}{|[x_i]^{\succ}_{R_1} \cap [x_i]^{\succ}_{R_2}|}} \leq 1.$$
Therefore,
$$0 \leq sim(R_1, R_2) \leq 1.$$

This completes the proof.

Property 3 Let $X = \{x_1, x_2, \cdots, x_n\}$ $(n > 1)$, R_1, R_2 be two non-empty rankings on X. If $R_1 = R_2$, then $sim(R_1, R_2) = 1$.

Proof: If $R_1 = R_2$, then for any $x_i \in X$, $[x_i]^{\succ}_{R_1} = [x_i]^{\succ}_{R_2}$ and $[x_i]^{\succ}_{R_1} \cup [x_i]^{\succ}_{R_2}$=$[x_i]^{\succ}_{R_1} \cap [x_i]^{\succ}_{R_2}$, so $|[x_i]^{\succ}_{R_1} \cup [x_i]^{\succ}_{R_2}|$=$|[x_i]^{\succ}_{R_1} \cap [x_i]^{\succ}_{R_2}|$, then,
$$\sum_{i=1}^{n} \log_2 \frac{n}{|[x_i]^{\succ}_{R_1} \cup [x_i]^{\succ}_{R_2}|} = \sum_{i=1}^{n} \log_2 \frac{n}{|[x_i]^{\succ}_{R_1} \cap [x_i]^{\succ}_{R_2}|}.$$
Hence, $sim(R_1, R_2) = 1$.

The following two properties show the characteristics about the similarity degree between two total rankings.

Property 4 Let $X = \{x_1, x_2, \cdots, x_n\}$ $(n > 1)$ and R be a total ranking on X. Without loss of generality, $R: \ x_1 \succ_R x_2 \succ_R \cdots \succ_R x_n$ and $R^{-1}: x_n \succ_{R^{-1}} x_{n-1} \succ_{R^{-1}} \cdots \succ_{R^{-1}} x_1$ is the reverse ordering of R, then $sim(R, R^{-1}) = 0$.

Proof: According to the orders of R and R^{-1}, $[x_i]^{\succ}_{R} = \{x_1, x_2, \cdots, x_i\}$, and $[x_i]^{\succ}_{R^{-1}} = \{x_i, x_{i+1}, \cdots, x_n\}, (i = 1, 2, \cdots, n)$. So we have that $|[x_i]^{\succ}_{R} \cup [x_i]^{\succ}_{R^{-1}}| = n$ and $|[x_i]^{\succ}_{R} \cap [x_i]^{\succ}_{R^{-1}}| = 1$. Therefore,
$$\begin{aligned} sim(R, R^{-1}) &= \frac{\sum\limits_{i=1}^{n} \log_2 \frac{n}{|[x_i]^{\succ}_{R} \cup [x_i]^{\succ}_{R^{-1}}|}}{\sum\limits_{i=1}^{n} \log_2 \frac{n}{|[x_i]^{\succ}_{R} \cap [x_i]^{\succ}_{R^{-1}}|}} \\ &= \frac{\sum\limits_{i=1}^{n} \log_2 \frac{n}{n}}{\sum\limits_{i=1}^{n} \log_2 n} \\ &= \frac{0}{n \log_2 n} \\ &= 0. \end{aligned}$$

Property 5 Let $X = \{x_1, x_2, \cdots, x_n\}$ $(n > 1)$. Let $R_1 : x_1 \succ_{R_1} \cdots \succ_{R_1} x_t \succ_{R_1} x_{t+1} \succ_{R_1} \cdots \succ_{R_1} x_n$ be a total ranking on X, $R_2 : x_1 \succ_{R_2} \cdots \succ_{R_2} x_{t+1} \succ_{R_2} x_t \succ_{R_2} \cdots \succ_{R_2} x_n$ be a total ranking by exchanging the positions of x_t and x_{t+1}, then

$$sim(R_1, R_2) = \frac{n \log_2 n - \log_2 \frac{(t+1)n!}{t}}{n \log_2 n - \log_2 \frac{t \times n!}{(t+1)}} \tag{5}$$

where $t \in \{1, 2, \cdots, (n-1)\}$.

Proof: For that

$$R_1 : x_1 \succ_{R_1} \cdots \succ_{R_1} x_t \succ_{R_1} x_{t+1} \succ_{R_1} \cdots \succ_{R_1} x_n$$

and

$$R_2 : x_1 \succ_{R_2} \cdots \succ_{R_2} x_{t+1} \succ_{R_2} x_t \succ_{R_2} \cdots \succ_{R_2} x_n.$$

We have

$$[x_j]^{\succ}_{R_1} = [x_j]^{\succ}_{R_2} = \{x_1, x_2, \cdots, x_j\} \ (j = 1, 2, \cdots, t-1, t+2, \cdots, n),$$

$$[x_t]^{\succ}_{R_1} = \{x_1, x_2, \cdots, x_t\}, \quad [x_{t+1}]^{\succ}_{R_1} = \{x_1, x_2, \cdots, x_{t+1}\},$$

$$[x_t]^{\succ}_{R_2} = \{x_2, x_2, \cdots, x_t, x_{t+1}\}, \quad [x_{t+1}]^{\succ}_{R_2} = \{x_1, x_2, \cdots, x_{t-1}, x_{t+1}\}.$$

Then,

$$|[x_j]^{\succ}_{R_1} \cap [x_j]^{\succ}_{R_2}| = |[x_j]^{\succ}_{R_1} \cup [x_j]^{\succ}_{R_2}| = j \ (j = 1, 2, \cdots, t-1, t+1, \cdots, n),$$

$$|[x_t]^{\succ}_{R_1} \cup [x_t]^{\succ}_{R_2}| = |\{x_1, x_2, \cdots, x_t, x_{t+1}\}| = t+1,$$

$$|[x_t]^{\succ}_{R_1} \cap [x_t]^{\succ}_{R_2}| = |\{x_1, x_2, \cdots, x_t\}| = t,$$

$$|[x_{t+1}]^{\succ}_{R_1} \cup [x_{t+1}]^{\succ}_{R_2}| = |\{x_1, x_2, \cdots, x_t, x_{t+1}\}| = t+1,$$

and

$$|[x_{t+1}]^{\succ}_{R_1} \cap [x_{t+1}]^{\succ}_{R_2}| = |\{x_1, x_2, \cdots, x_{t-1}, x_{t+1}\}| = t.$$

$$
\begin{aligned}
& sim(R_1,R_2) \\
= & \frac{\sum_{i=1}^{n}\log_2\frac{n}{|[x_i]^{\succ}_{R_1}\cup[x_i]^{\succ}_{R_2}|}}{\sum_{i=1}^{n}\log_2\frac{n}{|[x_i]^{\succ}_{R_1}\cap[x_i]^{\succ}_{R_2}|}} \\
= & \frac{\sum_{i=1}^{t-1}\log_2\frac{n}{i}+\sum_{i=t+2}^{n}\log_2\frac{n}{i}+\log_2\frac{n}{|[x_t]^{\succ}_{R_1}\cup[x_t]^{\succ}_{R_2}|}+\log_2\frac{n}{|[x_{t+1}]^{\succ}_{R_1}\cup[x_{t+1}]^{\succ}_{R_2}|}}{\sum_{i=1}^{t-1}\log_2\frac{n}{i}+\sum_{i=t+2}^{n}\log_2\frac{n}{i}+\log_2\frac{n}{|[x_t]^{\succ}_{R_1}\cap[x_t]^{\succ}_{R_2}|}+\log_2\frac{n}{|[x_{t+1}]^{\succ}_{R_1}\cap[x_{t+1}]^{\succ}_{R_2}|}} \\
= & \frac{\sum_{i=1}^{t-1}\log_2\frac{n}{i}+\sum_{i=t+2}^{n}\log_2\frac{n}{i}+\log_2\frac{n}{t+1}+\log_2\frac{n}{t+1}}{\sum_{i=1}^{t-1}\log_2\frac{n}{i}+\sum_{i=t+2}^{n}\log_2\frac{n}{i}+\log_2\frac{n}{t}+\log_2\frac{n}{t}} \\
= & \frac{(n-2)\log_2 n-\log_2\frac{n!}{t\times(t+1)}+\log_2\frac{n}{t+1}+\log_2\frac{n}{t+1}}{(n-2)\log_2 n-\log_2\frac{n!}{t\times(t+1)}+\log_2\frac{n}{t}+\log_2\frac{n}{t}} \\
= & \frac{n\log_2 n-\log_2\frac{(t+1)n!}{t}}{n\log_2 n-\log_2\frac{t\times n!}{t+1}}.
\end{aligned}
$$

This completes the proof.

Remark 2 Let $f(t)=\dfrac{n\log_2 n-\log_2\frac{(t+1)n!}{t}}{n\log_2 n-\log_2\frac{t\times n!}{t+1}}$,

$$
\begin{aligned}
& f'(t) \\
= & \frac{\left(n\log_2 n-\log_2\frac{(t+1)n!}{t}\right)'\left(n\log_2 n-\log_2\frac{t\times n!}{(t+1)}\right)-\left(n\log_2 n-\log_2\frac{t\times n!}{(t+1)}\right)'\left(n\log_2 n-\log_2\frac{(t+1)n!}{t}\right)}{\left(n\log_2 n-\log_2\frac{t\times n!}{(t+1)}\right)^2} \\
= & \frac{\left(\frac{\ln 2}{t\times(t+1)}\right)\left(n\log_2 n-\log_2\frac{t\times n!}{(t+1)}\right)+\left(\frac{\ln 2}{t\times(t+1)}\right)\left(n\log_2 n-\log_2\frac{(t+1)\times n!}{t}\right)}{\left(n\log_2 n-\log_2\frac{t\times n!}{(t+1)}\right)^2} \\
> & 0.
\end{aligned}
$$

Therefore, $f(t)$ is an increasing function. The bigger the subscript t is, the larger $sim(R_1,R_2)$ is. The result means that when we exchange the positions of the two objects in front of the total ranking, the similarity degree between the former and the latter is smaller, and the smaller similarity tells that the difference between the two total rankings is larger. Yet, we exchange the position of the two objects at the end of the total ranking, the similarity degree is larger and the difference is smaller. This property of similarity degree is coincide with the agreement index given in [21].

Example 4 Let $X=\{x_1,x_2,x_3,x_4\}$. $R_1: x_1\succ x_2\succ x_4\succ x_3$, $R_2: x_1\succ x_2\succ x_3\succ x_4$ and $R_3: x_2\succ x_1\succ x_3\succ x_4$ are three strict total rankings on X. We use Equation (4) to compute the similarity degree between each pair of rankings. Exchanging the positions of x_3 and x_4 in R_1, R_2 is obtained and $sim(R_1,R_2)=0.7833$. Exchanging the positions of x_1 and x_2 in R_2, R_3 is obtained and $sim(R_2,R_3)=0.5470$. According to the analysis and Property 5, the similarity between R_1 and R_3 should be the smallest, in fact, $sim(R_1,R_3) = 0.4141$.

Example 5 Suppose there is a set of four objects $X = \{x_1,x_2,x_3,x_4\}$. Three decision makers e_1, e_2 and e_3 evaluate the four objects and give their opinion as three rankings $R_1 : x_1 \succ x_2 \succ x_3 \succ x_4$, $R_2 : x_1 \succ x_2$ (x_3,x_4 are not compared), $R_3 : x_1 \succ x_3$ (x_2,x_4 are not compared). Using Equation (4), we have $sim(R_1,R_2) = 0.4879$, $sim(R_2,R_3) = 0.75$.

As shown in the calculated results, $sim(R_1,R_2) < sim(R_2,R_3)$. However, one may find that there exists the same comparison information $x_1 \succ x_2$ between the opinions of e_1 and e_2 and there is no common comparison information between e_2 and e_3, so it is may reasonable to get the reverse result $sim(R_1,R_2) > sim(R_2,R_3)$.

Remark 3 The properties and examples given above show that the similarity degree defined above is suitable for measuring the differences between two strict rankings from their mathematical structures. When the valid evaluation sets of the two rankings are different from each other, the similarity degree fails to work. In the following subsection, a new directional similarity degree, support degree, is presented to depict the closeness between two ranking judgments.

Definition 8 Let $X = \{x_1,x_2,\cdots,x_n\}$ $(n > 1)$. Let R_1, R_2 be two non-empty rankings on X. The support degree of R_2 from R_1 is defined as

$$sup(R_1,R_2) = \frac{|X_1 \cap X_2|}{|X_1|} sim_{X_1 \cap X_2}(R_1', R_2'), \tag{6}$$

where $X_1 = VER(R_1)$, $X_2 = VER(R_2)$, $R_1' = Proj_{X_1 \cap X_2}(R_1)$ and $R_2' = Proj_{X_1 \cap X_2}(R_2)$.

The support degree is not symmetric for that $\frac{|X_1 \cap X_2|}{|X_1|} \neq \frac{|X_1 \cap X_2|}{|X_2|}$ when $X_1 \neq X_2$.

Example 6 (Continued from Example 5.) $X_1 = VES(R_1) = \{x_1,x_2,x_3,x_4\}$, $X_2 = VES(R_2) = \{x_1,x_2\}$ and $X_3 = VES(R_3) = \{x_1,x_3\}$. $R_1' = Proj_{X_1 \cap X_2}(R_1) = Proj_{X_2}(R_1) : x_1 \succ x_2$, $R_2' = Proj_{X_1 \cap X_2}(R_2) = Proj_{X_2}(R_2) : x_1 \succ x_2$ Then,

$$\begin{aligned} sup(R_1,R_2) &= \tfrac{|X_1 \cap X_2|}{|X_1|} sim_{X_2}(Proj_{X_2}(R_1), Proj_{X_2}(R_2)) \\ &= \tfrac{1}{2} \times sim_{X_2}(R_1', R_2') \\ &= \tfrac{1}{2}, \\ sup(R_2,R_1) &= \tfrac{|X_1 \cap X_2|}{|X_2|} sim_{X_2}(Proj_{X_2}(R_1), Proj_{X_2}(R_2)) \\ &= sim_{X_2}(R_1', R_2') \\ &= 1. \end{aligned}$$

Similarly, $sup(R_1,R_3) = \frac{1}{2}$, $sup(R_3,R_1) = 1$, and $sup(R_2,R_3) = sup(R_3,R_2) = 0$.

Example 7 Let $X = \{x_1,x_2,x_3,x_4\}$. $R_1 : x_1 \succ x_2 \succ x_3 \succ x_4$, $R_2 : x_1 \succ x_2 \succ x_3$ (x_4 is not compared with the others) and $R_3 : x_1 \succ x_2$ (x_3 and x_4 are not compared with the others) are three rankings on X. We get $sup(R_1,R_2) = \frac{3}{4}$, $sup(R_1,R_3) = \frac{1}{2}$, and $sup(R_2,R_3) = \frac{2}{3}$. These results are much more closer to our rational judgment.

Property 6 Let $X = \{x_1, x_2, \cdots, x_n\}$ $(n > 1)$. Let R_1 and R_2 be two rankings on X. Then,

$$0 \leq sup(R_1, R_2) \leq 1.$$

Proof: For any R_1 and R_2, we have $0 \leq sim_{X_1 \cap X_2}(R_1, R_2) \leq 1$. Obviously, $0 \leq \frac{|VES(R_1) \cap VES(R_2)|}{|VES(R_1)|} \leq 1$, thus, $0 \leq sup(R_1, R_2) \leq 1$.

Property 7 Let $X = \{x_1, x_2, \cdots, x_n\}$ $(n > 1)$. Let R_1 and R_2 be two rankings on X. If $VES(R_1) = VES(R_2) = A$, then

$$sup(R_1, R_2) = sup(R_2, R_1) = sim_A(Proj_A(R_1), Proj_A(R_2)).$$

proof: It can be easily followed from the fact that

$$\frac{|VES(R_1) \cap VES(R_2)|}{|VES(R_1)|} = \frac{|VES(R_1) \cap VES(R_2)|}{|VES(R_2)|} = 1$$

and the symmetrical of the similarity degree.

Specially, if both of R_1 and R_2 are two total rankings on X, then $sup(R_1, R_2) = sim_X(R_1, R_2)$, i.e. the support degree is equal to the similarity degree. Property 7 shows that support degree is symmetrical under some special circumstances.

A decision maker's power or influence depends on the support degree and recognitions by the other decision makers in the real world. In [45], Yager gave the definition of support degree of one number from another numbers on a real number set and determine the corresponding average weight of each number based on the support degree. In what follows, we determine the weights of decision makers by the support degree.

Let $X = \{x_1, x_2, \cdots, x_n\}$ $(n > 1)$. Suppose e_i $(i = 1, 2, \ldots, l)$ expresses the judgment as a ranking $R_i \neq \emptyset$ $(i = 1, 2, \ldots, l)$. Let $T_i = \sum_{j=1, j \neq i}^{l} sup(R_j, R_i)$. Then, the weight of e_i is computed as

$$\eta_i = \frac{1 + T_i}{\sum_{j=1}^{n} (1 + T_j)}. \tag{7}$$

Actually, we can calculate the weights of decision makers from the support degree matrix directly. Let

$$M_{sup} = \begin{pmatrix} sup(R_1, R_1) & sup(R_1, R_2) & \ldots & sup(R_1, R_l) \\ sup(R_2, R_1) & sup(R_2, R_2) & \ldots & sup(R_2, R_l) \\ \vdots & \vdots & & \vdots \\ sup(R_l, R_1) & sup(R_l, R_2) & \ldots & sup(R_l, R_l) \end{pmatrix}. \tag{8}$$

Then, η_i is the normalization of the sum of the ith column for that $sup(R_i, R_i) = 1$. Since η_i $(i = 1, 2, \ldots, l)$ is calculated from the relations of granular structures and not given by the decision makers, it is a kind of objective weights.

Remark 4 In practical application, the final weights of decision makers might be calculated by integrating the objective weights and the subjective ones. Here, we consider the objective weights only.

4 An improved programming model for weighted ordinal rankings

Suppose decision makers $e_1, e_2, \ldots, e_l$ express their judgments as l strict non-empty rankings $R_1, R_2, \ldots, R_l$ on a set of objects $X = \{x_1, x_2, \ldots, x_n\}$. We can calculate the objective weights of the l decision makers as $\eta_1, \eta_2, \ldots, \eta_l$ by using the Eqs. (4)-(8) in Section 3. Then we construct the violation matrix as

$$V' = (v'_{ij})_{n\times n},$$

where $v'_{ij} = \sum_{\substack{k=1 \\ x_j \succ_{R_k} x_i}}^{l} \eta_k$ and η_k is the weight of the k^{th} DMs.

The mathematical programming, Eq. (3), should be changed in order to solve the following weighted GDM problems

$$\min_{R\in\mathbb{R}^X} \mathfrak{M}'(R) = \sum_{x_i \succ_R x_j} v'_{ij}. \tag{9}$$

The meaning of v'_{ij} is different from v_{ij} in Eq. (3).

If there are a few objects (such as less than 8) in the object set, we can use the enumeration method to get the optimal solution in an acceptable computation time, otherwise, the Branch-and-Bound method [13] can be applied to solve Eq. (9).

In reality, different decision makers have different judgement abilities, and their judgments may influence the decision making result differently, so it is reasonable to set different weights for different decision makers in a credible decision making process. However, the Cook's model grants each decision maker with the equal weight, which usually does not conform to the reality. Considering the decision makers' weights, we have proposed a new model to calculate the total ranking(s) by minimizing the weighted violation. In the new model, if a violation is provided by a decision maker with a smaller weight, then it contributes less to the objective; if a violation comes from a decision maker with a higher weight, it contributes more to the objective.

5 Illustrative examples

In this section, two GDM examples are given to illustrate the rationality of the proposed weighted ordinal ranking model.

5.1 The first numerical example

Four objects x_1, x_2, x_3, x_4 are evaluated by four DMs e_1, e_2, e_3, e_4 with the ordinal ranking expression. The ordinal rankings are expressed in Table 2. The task is to derive the total ranking(s) from the total and partial rankings with the minimum violation.

Table 2 The first partial ranking expressed decision making problem

DM	Evaluated objects	Partial Ranking
e_1	$\{x_1,x_2,x_3,x_4\}$	$R_1: x_1 \succ x_2 \succ x_3 \succ x_4$
e_2	$\{x_1,x_2\}$	$R_2: x_1 \succ x_2$ (x_3,x_4 *are not compared*)
e_3	$\{x_3,x_4\}$	$R_3: x_3 \succ x_4$ (x_1,x_2 *are not compared*)
e_4	$\{x_1,x_2,x_3,x_4\}$	$R_4: x_4 \succ x_3 \succ x_2 \succ x_1$

(1) Determine the weights of DMs. Take the calculation of $sup(R_1,R_3)$ for example, by using Eqs.(4) and (6),

$$\begin{aligned} sup(R_1,R_3) &= \frac{|X_1 \cap X_3|}{|X_1|} sim_{X_1\cap X_3}(Proj_{X_1\cap X_3}(R_1), Proj_{X_1\cap X_3}(R_3)) \\ &= \frac{1}{2} \times sim_{X_3}(x_3 \succ x_4, x_3 \succ x_4) \\ &= 0.5, \end{aligned}$$

where $X_1 = \{x_1,x_2,x_3,x_4\}$ and $X_3 = \{x_3,x_4\}$. Then, the support degree matrix is computed as

$$M_{sup} = (sup(R_i,R_j)) = \begin{pmatrix} 1 & 0.5 & 0.5 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Eq. (7), we calculate the weights of the four decision makers from the support degree matrix as $\eta_1 = \frac{1+T_i}{\sum_{j=1}^{4}(1+T_j)} = \frac{\sum_{j=1}^{4} sup(R_j,R_1)}{\sum_{i=1}^{4}\sum_{j=1}^{4} sup(R_j,R_i)} = 0.4286$, similarly, $\eta_2 = 0.2143, \eta_3 = 0.2143, \eta_4 = 0.1429$.

(2) Construct the violation matrix $(r'_{ij})_{4\times 4}$. For example, $r'_{43} = \eta_1 + \eta_3 = 0.6538$, since e_1 and e_3 consider that x_3 dominates x_4. The others follow in a similar manner. Then,

$$\begin{aligned} (r'_{ij})_{4\times 4} &= \begin{pmatrix} 0 & \eta_4 & \eta_4 & \eta_4 \\ \eta_1+\eta_2 & 0 & \eta_4 & \eta_4 \\ \eta_1 & \eta_1 & 0 & \eta_4 \\ \eta_1 & \eta_1 & \eta_1+\eta_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0.1429 & 0.1429 & 0.1429 \\ 0.6429 & 0 & 0.1429 & 0.1429 \\ 0.4286 & 0.4286 & 0 & 0.1429 \\ 0.4286 & 0.4286 & 0.6429 & 0 \end{pmatrix}. \end{aligned}$$

(3) Solve the programming problem, Eq. (9), with the violation matrix $(r'_{ij})_{4\times 4}$. By using the enumeration method, we get the optimal solution as $x_1 \succ x_2 \succ x_3 \succ x_4$.

If we did not consider the weights of the decision makers, we would get tied optimal consensus rankings $x_1 \succ x_2 \succ x_3 \succ x_4$ and $x_1 \succ x_3 \succ x_2 \succ x_4$. In our method, weights of the four decision makers are calculated and a new violation matrix is construct to calculate the result $x_1 \succ x_2 \succ x_3 \succ x_4$, which is more reasonable than $x_1 \succ x_3 \succ x_2 \succ x_4$. Since most of decision makers support the judgment of e_1, and e_1 prefer x_2 to x_3, x_2 should be preferred to x_3.

Table 3 The second partial ranking expressed decision making problem

DM	Evaluated objects	Partial Ranking
e_1	$\{x_1,x_2,x_3,x_5\}$	$R_1: x_1 \succ x_3 \succ x_2 \succ x_5$ $(x_4,x_6$ *are not compared*$)$
e_2	$\{x_1,x_2,x_4,x_6\}$	$R_2: x_2 \succ x_1 \succ x_4 \succ x_6$ $(x_5,x_6$ *are not compared*$)$
e_3	$\{x_3,x_4,x_5,x_6\}$	$R_3: x_4 \succ x_3 \succ x_5 \succ x_6$ $(x_1,x_2$ *are not compared*$)$
e_4	$\{x_1,x_4,x_5,x_6\}$	$R_4: x_6 \succ x_1 \succ x_4 \succ x_5$ $(x_2,x_3$ *are not compared*$)$
e_5	$\{x_1,x_2,x_3,x_6\}$	$R_5: x_6 \succ x_2 \succ x_3 \succ x_1$ $(x_4,x_5$ *are not compared*$)$

5.2 The second numerical example

This example is chosen from Ref. [13]. Six objects x_1,x_2,x_3,x_4,x_5,x_6 are evaluated by five DMs e_1,e_2,e_3,e_4,e_5. The strict partial ordinal rankings are expressed in Table 3. It is our task to derive the total ranking(s) based on the partial rankings provided by the five DMs.

(1) Determine the weights of the decision makers.

First of all, compute the support degree matrix by Eqs. (4) and (6) as

$$M_{sup} = (sup(R_i,R_j)) = \begin{pmatrix} 1 & 0 & 0.5 & 0.5 & 0 \\ 0 & 1 & 0.5 & 0.1169 & 0.1169 \\ 0.5 & 0.5 & 1 & 0.1169 & 0 \\ 0.5 & 0.1169 & 0.1169 & 1 & 0.5 \\ 0 & 0.1169 & 0 & 0.5 & 1 \end{pmatrix}.$$

Take the calculation of $sup(R_2,R_4)$ for example,

$$\begin{aligned} sup(R_2,R_4) &= \tfrac{|X_2 \cap X_4|}{|X_2|} sim\,(Proj_{X_2\cap X_4}(R_2), Proj_{X_2\cap X_4}(R_4)) \\ &= \tfrac{3}{4} \times sim_{X_2\cap X_4}(x_1 \succ x_4 \succ x_6, x_6 \succ x_1 \succ x_4) \\ &= 0.1169, \end{aligned}$$

where $X_2 = \{x_1,x_2,x_4,x_6\}$ and $X_4 = \{x_1,x_4,x_5,x_6\}$. By using Eq. (7), the weights of the decision makers are calculated as $\eta_1 = 0.2062,\ \eta_2 = 0.1787,\ \eta_3 = 0.2182,\ \eta_4 = 0.2303,\ \eta_5 = 0.1667$.

(2) Build the violation matrix $(r'_{ij})_{6\times6}$. For example, $v'_{12} = \eta_2 + \eta_5 = 0.1787 + 0.1667 = 0.3454$, that is because e_2 and e_5 consider that x_2 dominates x_1. Then the violation matrix is computed as

$$\begin{aligned} (r'_{ij})_{6\times6} &= \begin{pmatrix} 0 & \eta_2+\eta_5 & \eta_5 & 0 & 0 & \eta_4+\eta_5 \\ \eta_1 & 0 & \eta_1 & 0 & 0 & \eta_5 \\ \eta_1 & \eta_5 & 0 & \eta_3 & 0 & \eta_5 \\ \eta_2+\eta_4 & \eta_2 & 0 & 0 & 0 & \eta_4 \\ \eta_1+\eta_4 & \eta_1 & \eta_1+\eta_3 & \eta_3+\eta_4 & 0 & \eta_4 \\ \eta_2 & \eta_2 & \eta_3 & \eta_2+\eta_3 & \eta_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0.3454 & 0.1667 & 0 & 0 & 0.3970 \\ 0.2062 & 0 & 0.2062 & 0 & 0 & 0.1667 \\ 0.2062 & 0.1667 & 0 & 0.2182 & 0 & 0.1667 \\ 0.409 & 0.1787 & 0 & 0 & 0 & 0.2303 \\ 0.4365 & 0.2062 & 0.4244 & 0.4485 & 0 & 0.2303 \\ 0.1787 & 0.1787 & 0.2182 & 0.3969 & 0.2182 & 0 \end{pmatrix}. \end{aligned}$$

(3) Solving Eq. (9) with the violation matrix $(r'_{ij})_{5\times5}$, we obtain the optimal consensus total ranking $x_2 \succ x_6 \succ x_1 \succ x_4 \succ x_3 \succ x_5$.

The optimal total ranking ($x_2 \succ x_6 \succ x_1 \succ x_4 \succ x_3 \succ x_5$) derived from our model is different from Cook's ($x_2 \succ x_1 \succ x_4 \succ x_6 \succ x_3 \succ x_5$). Due to the decision makers e_4 and e_5 with the higher weights support x_6 and place it in the first place, then x_6 takes the second place in the final ranking.

It can be seen from the two examples that considering the objective weights of decision makers which computed from the structures of their judgments would bring out different results from the ones with equal weights and the decision making results computed from the former are more reasonable than the latter.

6 Conclusions

The ranking is a common-used form of preference representation in a range of practical situations. Some researchers have successfully solved the problem about deriving consensus total ranking(s) from an equal weighted ranking set. In this study, based on the support degree of each pair of rankings, an objective weighting method is presented to determine the relative importance of the decision makers and an improved programming model is proposed to derive the consensus ranking(s) from the weighted ranking set.

The current study develops the method for determining the weights of decision makers and derives the minimum violation consensus ranking(s) form a set of weighted partial ordinal rankings. There are mainly two interesting topics to be investigated in our future research. One is to determine the objective weights of decision makers which express their judgments by preorders. The other is to design a network based decision making system to aid the distributed group decision making.

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