Information Sciences 179 (2009) 2809–2832

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Set-valued ordered information systems

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ARTICLE INFO

Article history:
Received 10 January 2008
Received in revised form 4 March 2009
Accepted 8 April 2009

Keywords:
Set-valued ordered information systems
Set-valued ordered decision tables
Dominance relation
Rough set
Criteria reduction

ABSTRACT

Set-valued ordered information systems can be classified into two categories: disjunctive and conjunctive systems. Through introducing two new dominance relations to set-valued information systems, we first introduce the conjunctive/disjunctive set-valued ordered information systems, and develop an approach to queuing problems for objects in presence of multiple attributes and criteria. Then, we present a dominance-based rough set approach for these two types of set-valued ordered information systems, which is mainly based on substitution of the indiscernibility relation by a dominance relation. Through the lower/upper approximation of a decision, some certain/possible decision rules from a so-called set-valued ordered decision table can be extracted. Finally, we present attribute reduction (also called criteria reduction in ordered information systems) approaches to these two types of ordered information systems and ordered decision tables, which can be used to simplify a set-valued ordered information system and find decision rules directly from a set-valued ordered decision table. These criteria reduction approaches can eliminate those criteria that are not essential from the viewpoint of the ordering of objects or decision rules.

1. Introduction

Rough set theory, proposed by Pawlak [30,31], has been conceived as a tool to conceptualize and analyze various types of data. It can be used in attribute value representation models to describe the dependencies among attributes, evaluate the significance of attributes and derive decision rules. The theory shows important applications to intelligent decision-making and cognitive sciences, as a tool for dealing with vagueness and uncertainty of information [4,5,7,14,20,25,46,49,51].

Originally, rough set theory is based on an assumption that every object in the universe of discourse is associated with some information. Objects characterized by the same information are indiscernible. The indiscernibility relation generated in this way forms the mathematical basis for the theory of rough sets. The set of all indiscernible objects is called an elementary set or equivalence class [24,41]. A rough set can be characterized by a pair of sets, called the lower and upper approximations. Rough set-based data analysis starts from a data table, also called an information system, which contains data about objects of interest that are characterized by a finite set of attributes [11,15,16,19,21,22,32–37,41]. In recent years, classical rough sets have been extended to several general models by using other binary relations, see [26,27,42–45].

However, the original rough set theory does not consider attributes with preference-ordered domains, that is, criteria. In fact, in many real-world situations, we are often faced with the problems in which the ordering of properties of the considered attributes plays a crucial role [1,13]. One such type of problem is the ordering of objects. For this reason, Greco et al. [8,9] proposed an extension of rough set theory, called dominance-based rough set approach (DRSA) to take into account

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doi:10.1016/j.ins.2009.04.007

the ordering properties of criteria. This generalization is mainly based on the substitution of the indiscernibility relation by a dominance relation. In DRSA, where condition attributes are criteria and classes are preference ordered, the knowledge approximated is a collection of upward and downward unions of classes and the granules of knowledge are sets of objects defined by using a dominance relation, see [2,3,39,40].

In what follows, we review several types of information systems. By an incomplete information system we mean a system with missing data (null values) [16,17]. Incomplete information systems deal with two cases: unknown values and inapplicable values. In unknown values, a null value may be some value in the domain of the corresponding attribute [16–18]; for the case that a null value means an inapplicable value, it can be handled by adding to the attribute domains a special symbol for the inapplicable value [18,37]. If the value of each object is represented as a certain fuzzy set, we are concerned with fuzzy information systems [12,14,42]. In the context of fuzzy information systems, fuzzy rough set approach has been developed [50,53,54]. Interval information systems are an important type of information systems, and generalized models of single-valued information systems. Some problems of decision making in the context of interval information systems have also been studied in [38,47,48], most of which are based on the concept of a possible degree between any two interval numbers [47,48]. It is often interesting to discover some dependency relationships (patterns) among attributes from these kinds of information systems [52,53].

In many practical issues, it may happen that some of the attribute values for an object are set-valued, which are always used to characterize uncertain information and missing information in information systems [52,53]. For example, in language-ability test information systems there may exist a group of candidates that can master several kinds of languages. These values can be represented by the set of these languages for the attribute. To describe such situation, a set value is usually assigned to those attributes. In general, this kind of information systems are called set-valued information systems, which are another important type of data tables and generalized models of single-valued information systems. For instance, incomplete information systems can be regarded as a special kind of set-valued information systems, in which all missing values can be represented by the set of all possible values of each attribute. In order to make a decision in set-valued information systems, it is interesting and desirable to investigate dominance relations, ranking approach, rough set framework, dominance rules and attribute reduction in this kind of information systems.

Let us introduce a formal definition of set-valued information systems. Let U be a finite set of objects, called the universe of discourse, and AT be a finite set of attributes. With every attribute $a \in AT$, a set of its values V_a is associated. $f : U \times AT \rightarrow V$ is a total function such that $f(x,a) \subseteq V_a$ for every $a \in AT, x \in U$. If each attribute has a unique attribute value, then (U, AT, V, f) with $V = \cup_{a \in AT} V_a$ is called a single-valued information system; if a system is not a single-valued information system, it is called a set-valued (multi-valued) information system. If we consider condition and decision attributes, then such an information system is called a set-valued decision information system. A set-valued decision information system is always denoted by $S = (U, C \cup \{d\}, V, f)$, where C is a finite set of condition attributes, and d is a decision attribute with $C \cap d = \emptyset$.

There are many ways to provide semantics of set-valued information systems [6,23,28,29], here we summarize two types of them [10]:

Type I: For $x \in U$ and $c \in C$, $c(x)$ is interpreted disjunctively. For example: If c is the attribute "speaking a language", then $c(x) = \{\text{German}, \text{Polish}, \text{French}\}$ can be interpreted as: x speaks German, Polish, or French, and x can speak only one of them. Incomplete information systems with some unknown attribute values or partial known attribute values [16–18] are such type of set-valued information systems. Under the consideration, we call it a "$\bigvee$" (disjunctive) set-valued information system in this paper.

Type II: For $x \in U$ and $c \in C$, $c(x)$ is interpreted conjunctively. For example: If c is the attribute "speaking a language", then $c(x) = \{\text{German}, \text{Polish}, \text{French}\}$ can be interpreted as: x speaks German, Polish, and French. When considering the attribute "feeding habits" of animals, if we denote the attribute value of herbivore as "0" and carnivore as "1", then animals possessing attribute value $\{0,1\}$ are considered as possessing both herbivorous and carnivorous nature. Let us take blood origin for another example, if we denote the three types of pure blood as "0", "1" and "2", then we can denote the mixed-blood as $\{0,1\}$ or $\{1,2\}$, etc. Under such interpretation, in this paper we call it a "$\bigwedge$" (conjunctive) set-valued information system.

The main objective of this study is to introduce two new dominance relations to the two types of set-valued information systems, and investigate the problems of criteria reductions and decision rules extracted from these two types of information systems and decision tables.

The paper is organized as follows: Some preliminary concepts about ordered information systems are briefly reviewed in Section 2. In Section 3, we introduce two dominance relations $R_A^{\wedge\geqslant}$ and $R_A^{\vee\geqslant}$ to "$\bigwedge$" set-valued information systems and "$\bigvee$" set-valued information systems, respectively, and present a ranking approach to all objects under the dominance relations. Based on these two dominance relations $R_A^{\wedge\geqslant}$ and $R_A^{\vee\geqslant}$, we establish a rough set approach in a set-valued ordered information system, and some of its important properties are obtained. In Section 4, decision rules from the two types of set-valued decision tables are discussed. In Section 5, we investigate the approaches to attribution reductions in these two types of set-valued ordered information systems and decision tables. Finally, conclusions are presented in Section 6.

2. Some basic concepts

An information system (IS) is an quadruple $S = (U, AT, V, f)$, where U is a finite nonempty set of objects and AT is a finite nonempty set of attributes, $V = \bigcup_{a \in AT} V_a$ and V_a is a domain of attribute a, $f : U \times AT \rightarrow V$ is a function such that $f(x,a) \in V_a$

for every $a \in AT, x \in U$, called an information function. A decision table is a special case of an information system in which, among the attributes, we distinguish one called a decision attribute. The other attributes are called condition attributes. Therefore, $S = (U, C \cup d, V, f)$ and $C \cap d = \emptyset$, where set C contains so-called condition attributes and d, the decision attribute.

If the domain (scale) of a condition attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.[1]

Definition 2.1 [40]. An information system is called an ordered information system (OIS) if all condition attributes are criteria.

It is assumed that the domain of a criterion $a \in AT$ is completely pre-ordered by an outranking relation $\succcurlyeq_a$; $x \succcurlyeq_a y$ means that x is at least as good as (outranks) y with respect to criterion a. In the following, without any loss of generality, we consider a condition criterion having a numerical domain, that is, $V_a \subseteq \mathbf{R}$ ($\mathbf{R}$ denotes the set of real numbers) and being of type gain, that is, $x \succcurlyeq_a y \Longleftrightarrow f(x,a) \geqslant f(y,a)$ (according to increasing preference) or $x \succcurlyeq_a y \Longleftrightarrow f(x,a) \leqslant f(y,a)$ (according to decreasing preference), where $a \in AT$, $x, y \in U$. For a subset of attributes $B \subseteq C$, we define $x \succcurlyeq_B y \Longleftrightarrow \forall a \in B, f(x,a) \geqslant f(y,a)$. In other words, x is at least as good as y with respect to all attributes in B. In general, the domain of the condition criterion may be also discrete, but the preference order between its values has to be provided.

In what follows, we review the dominance relation that identifies granules of knowledge. In a given OIS, we say that x dominates y with respect to $B \subseteq C$ if $x \succcurlyeq_B y$, and denoted by $xR_B^{\geqslant}y$. That is

$$R_B^{\geqslant} = \{(y,x) \in U \times U \mid y \succcurlyeq_B x\}.$$

Obviously, if $(y,x) \in R_B^{\geqslant}$, then y dominates x with respect to B.

Let B_1 be attributes set according to increasing preference, B_2 attributes set according to decreasing preference, hence $B = B_1 \cup B_2$. The granules of knowledge induced by the dominance relation $R_B^{\geqslant}$ are the set of objects dominating x, i.e.,

$$[x]_B^{\geqslant} = \{y \in U | f(y,a_1) \geqslant f(x,a_1)(\forall a_1 \in B_1) \text{ and } f(y,a_2) \leqslant f(x,a_2)(\forall a_2 \in B_2)\} = \{y \in U | (y,x) \in R_B^{\geqslant}\}$$

and the set of objects dominated by x,

$$[x]_B^{\leqslant} = \{y \in U | f(y,a_1) \leqslant f(x,a_1)(\forall a_1 \in B_1) \text{ and } f(y,a_2) \geqslant f(x,a_2)(\forall a_2 \in B_2)\} = \{y \in U | (x,y) \in R_B^{\geqslant}\},$$

which are called the B-dominating set and the B-dominated set with respect to $x \in U$, respectively.

Let $U/R_B^{\geqslant}$ denote classification on the universe, which is the family set $\{[x]_B^{\geqslant} | x \in U\}$. Any element from $U/R_B^{\geqslant}$ will be called a dominance class with respect to B. Dominance classes in $U/R_B^{\geqslant}$ do not constitute a partition of U in general. They constitute a covering of U.

For simplicity and without any loss of generality, in the following we only consider condition attributes with increasing preference.

The following property can be easily concluded [40].

Property 2.1. *Let $R_B^{\geqslant}$ be a dominance relation, then*

(1) $R_B^{\geqslant}$ is reflexive, transitive and unsymmetric, so it is not an equivalence relation;
(2) if $A \subseteq B \subseteq C$, then $R_C^{\geqslant} \subseteq R_B^{\geqslant} \subseteq R_A^{\geqslant}$;
(3) if $A \subseteq B \subseteq C$, then $[x]_C^{\geqslant} \subseteq [x]_B^{\geqslant} \subseteq [x]_A^{\geqslant}$;
(4) if $x_j \in [x_i]_B^{\geqslant}$, then $[x_j]_B^{\geqslant} \subseteq [x_i]_B^{\geqslant}$ and $[x_i]_B^{\geqslant} = \bigcup\{[x_j]_B^{\geqslant} : x_j \in [x_i]_B^{\geqslant}\}$;
(5) $[x_i]_B^{\geqslant} = [x_j]_B^{\geqslant}$ iff $f(x_i,a) = f(x_j,a)(\forall a \in B)$;
(6) $\mathsf{F} = \{[x]_B^{\geqslant} | x \in U\}$ constitutes a covering of U.

For any $X \subseteq U$ and $B \subseteq C$, the lower and upper approximation of X with respect to the dominance relation $R_B^{\geqslant}$ are defined as follows

$$\underline{R_B^{\geqslant}}(X) = \{x \in U | [x]_B^{\geqslant} \subseteq X\},$$
$$\overline{R_B^{\geqslant}}(X) = \{x \in U | [x]_B^{\geqslant} \cap X \neq \emptyset\}.$$

Unlike classical rough set theory, one can easily notice that the properties $\underline{R_B^{\geqslant}}(X) = \bigcup\{[x]_B^{\geqslant} | [x]_B^{\geqslant} \subseteq X\}$ and $\overline{R_B^{\geqslant}}(X) = \bigcup\{[x]_B^{\geqslant} | [x]_B^{\geqslant} \cap X \neq \emptyset\}$ do not hold.

Example 2.1. An OIS is presented in Table 1, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $AT = \{a_1, a_2, a_3\}$.

The dominance classes determined by AT are

$$[x_1]_{AT}^{\geqslant} = \{x_1, x_2, x_5, x_6\}, \quad [x_2]_{AT}^{\geqslant} = \{x_2, x_5, x_6\}, \quad [x_3]_{AT}^{\geqslant} = \{x_2, x_3, x_4, x_5, x_6\}, \quad [x_4]_{AT}^{\geqslant} = \{x_4, x_6\},$$
$$[x_5]_{AT}^{\geqslant} = \{x_5\}, \quad [x_6]_{AT}^{\geqslant} = \{x_6\}.$$

[1] In intelligent decision-making, each of attributes in ordered information systems and ordered decision tables are always called a criterion, which is mainly used to rank for all objects and to choose the best project from all projects [8,38,40].

Table 1
An ordered information system.

U	a_1	a_2	a_3
x_1	1	2	1
x_2	3	2	2
x_3	1	1	2
x_4	2	1	3
x_5	3	3	2
x_6	3	2	3

Suppose that $X = \{x_2, x_3, x_5\}$, then

$$\underline{R^{\geqslant}_{AT}}(X) = \{x_5\} \subseteq X, \quad \overline{R^{\geqslant}_{AT}}(X) = \{x_1, x_2, x_3, x_5\} \supseteq X.$$

3. Set-valued ordered information systems

In some practical issues, it may happen that some of the attribute values of an object are set-valued. Therefore, a so-called set-valued information system, is usually used to indicate such a situation. Let $S = (U, AT, V, f)$ be a set-valued information system, where U is a nonempty finite set of objects, AT is a finite set of attributes, V is the set of attributes values and f is a mapping from $U \times AT$ to V such that $f : U \times AT \to 2^V$ is a set-valued mapping. In this situation, the cardinality $|f(x, a)| \geqslant 1, \ \forall x \in U, a \in AT$. In this section, through introducing two dominance relations to two types of set-valued information systems, we investigate conjunctive set-valued ordered information systems and disjunctive set-valued ordered information systems, and propose a ranking method for all objects and a rough set approach to these two particular systems.

3.1. Conjunctive set-valued ordered information systems

In this subsection, we will deal with conjunctive set-valued ordered information systems and discuss some of their important properties. The following example presents a conjunctive set-valued information system.

Example 3.1. A set-valued information system is presented in Table 2, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $AT = \{a_1, a_2, a_3, a_4\} = \{\text{Audition}, \text{Spoken language}, \text{Reading}, \text{Writing}\}$ and $V = \{\text{English}, \text{French}, \text{German}\}$. For convenience, in the sequel, E, F and G will stand for English, French and German, respectively.

From Table 2, it is easy to see that the values in arbitrary set-value $f(x_i, a)$ $(i \leqslant 10, \ a \in AT)$ are all discrete. Unlike for some existing dominance relations, we can not say that E is at least as good as F or E is at most as good as F in this type of information systems. If we regard a conjunctive type of interpretation, then Table 2 is a "$\bigwedge$" set-valued information system.

In Table 2, we can know that $f(x_7, a_1) = \{E, F, G\}$, $f(x_8, a_1) = \{E, F\}$. Because of $\{E, F, G\} \supseteq \{E, F\}$, we can judge that the audition ability of x_7 must be better than that of x_8, that is, x_7 is at least as good as x_8 with respect to a_1. We call this type of preference *inclusion increasing preference*. In addition, for instance, in a conjunctive set-valued information system about diagnosing rheum, if the patient x has three characters $\{Headache, Musclepain, fever\}$ and the patient y has two characters $\{Headache, fever\}$, we may say that the patient y is better than the patient x with respect to *Rheum characters*. In other words, y is at least as good as x with respect to *Rheum characters*. We call this type of preference *inclusion decreasing preference*.

If the values of some objects under a condition attribute can be ordered according to an inclusion increasing/decreasing preference, then the attribute is a *inclusion criterion*.

Definition 3.1. A conjunctive set-valued information system is called a conjunctive *set-valued ordered information system* (OIS) if all condition attributes are inclusion criterions.

Table 2
A set-valued information system about language ability.

U	Audition	Spoken language	Reading	Writing
x_1	$\{E\}$	$\{E\}$	$\{F, G\}$	$\{F, G\}$
x_2	$\{E, F, G\}$	$\{E, F, G\}$	$\{F, G\}$	$\{E, F, G\}$
x_3	$\{E, G\}$	$\{E, F\}$	$\{F, G\}$	$\{F, G\}$
x_4	$\{E, F\}$	$\{E, G\}$	$\{F, G\}$	$\{F\}$
x_5	$\{F, G\}$	$\{F, G\}$	$\{F, G\}$	$\{F\}$
x_6	$\{F\}$	$\{F\}$	$\{E, F\}$	$\{E, F\}$
x_7	$\{E, F, G\}$	$\{E, F, G\}$	$\{E, G\}$	$\{E, F, G\}$
x_8	$\{E, F\}$	$\{F, G\}$	$\{E, F, G\}$	$\{E, G\}$
x_9	$\{F, G\}$	$\{G\}$	$\{F, G\}$	$\{F, G\}$
x_{10}	$\{E, F\}$	$\{E, G\}$	$\{F, G\}$	$\{E, F\}$

It is assumed that the domain of an inclusion criterion $a \in AT$ is completely pre-ordered by an outranking relation $\succcurlyeq_a$; $x \succcurlyeq_a y$ means that x is at least as good as (outranks) y with respect to the inclusion criterion a. In the following, without any loss of generality, we consider an inclusion criterion having an enumerative domain, that is, $x \succcurlyeq_a y \Longleftrightarrow f(x,a) \supseteq f(y,a)$ (according to inclusion increasing preference) or $x \succcurlyeq_a y \Longleftrightarrow f(x,a) \subseteq f(y,a)$ (according to inclusion decreasing preference), where $a \in AT$ and $x, y \in U$. For a subset of attributes $B \subseteq C$, we define $x \succcurlyeq_B y \Longleftrightarrow \forall a \in B, f(x,a) \supseteq f(y,a)$. In other words, x is at least as good as y with respect to all attributes in B.

For a "$\bigwedge$" set-valued information system $S = (U, AT, V, f)$, the relationships among any set $f(x,a), x \in U, a \in AT$ are conjunctive. While they are disjunctive in a "$\bigvee$" set-valued information system. Hence, for convenience, let $R_A^{\wedge\geqslant}$, $A \subseteq AT$, denote a binary dominance relation between objects that are possibly dominant in terms of values of attributes set A (it is denoted by $R_A^{\vee\geqslant}$ in "$\bigvee$" set-valued information systems). Under this consideration, S is a conjunctive set-valued ordered information system. Let us define the dominance relation more precisely as follows

$$R_A^{\wedge\geqslant} = \{(y,x) \in U \times U | f(y,a) \supseteq f(x,a)(\forall a \in A_1) \quad \text{and} \quad f(y,a) \subseteq f(x,a)(\forall a \in A_2)\} = \{(y,x) \in U \times U | y \succcurlyeq_A x\},$$

where A_1 is attributes set according to inclusion increasing preference, A_2 is attributes set according to inclusion decreasing preference and $A = A_1 \cup A_2$.

By the definition of the dominance relation $R_A^{\wedge\geqslant}$, it can be observed that if a pair of objects (y,x) from $U \times U$ lies in $R_A^{\wedge\geqslant}$, then they are perceived as y dominates x; in other words, y may have a better property than x with respect to A in reality.

Analogously, the relation $R_A^{\wedge\leqslant}$ can be defined as follows

$$R_A^{\wedge\leqslant} = \{(y,x) \in U \times U | f(y,a) \subseteq f(x,a)(\forall a \in A_1) \text{ and } f(y,a) \supseteq f(x,a)(\forall a \in A_2)\} = \{(y,x) \in U \times U | x \succcurlyeq_A y\},$$

From the definition of $R_A^{\wedge\geqslant}$ and $R_A^{\wedge\leqslant}$, the following properties can be easily obtained.

Property 3.1. *Let $S = (U, AT, V, f)$ be a conjunctive set-valued ordered information system and $A \subseteq AT$, then*

$$R_A^{\wedge\geqslant} = \bigcap_{a\in A} R_a^{\wedge\geqslant}, \quad R_A^{\wedge\leqslant} = \bigcap_{a\in A} R_a^{\wedge\leqslant}.$$

Property 3.2. *Let $R_A^{\wedge\geqslant}$ be an inclusion dominance relation in a conjunctive set-valued information system, then*

(1) $R_A^{\wedge\geqslant}$ is reflexive;
(2) $R_A^{\wedge\geqslant}$ is unsymmetric;
(3) $R_A^{\wedge\geqslant}$ is transitive.

Furthermore, denoted by

$$[x]_A^{\wedge\geqslant} = (\{y \in U | (y,x) \in R_A^{\wedge\geqslant}\},$$
$$[x]_A^{\wedge\leqslant} = \{y \in U | (x,y) \in R_A^{\wedge\geqslant}\},$$

where $[x]_A^{\wedge\geqslant}$ describes objects that may dominate x and $[x]_A^{\wedge\leqslant}$ describes objects that may be dominated by x in terms of A in a conjunctive set-valued ordered information system.

Property 3.3. *Let $S = (U, AT, V, f)$ be a conjunctive set-valued ordered information system and $A, B \subseteq AT$, one has*

(1) if $B \subseteq A \subseteq AT$, then $R_B^{\wedge\geqslant} \supseteq R_A^{\wedge\geqslant} \supseteq R_{AT}^{\wedge\geqslant}$;
(2) if $B \subseteq A \subseteq AT$, then $[x]_B^{\wedge\geqslant} \supseteq [x]_A^{\wedge\geqslant} \supseteq [x]_{AT}^{\wedge\geqslant}$;
(3) if $x_j \in [x_i]_A^{\wedge\geqslant}$, then $[x_j]_A^{\wedge\geqslant} \subseteq [x_i]_A^{\wedge\geqslant}$ and $[x_i]_A^{\wedge\geqslant} = \bigcup\{[x_j]_A^{\wedge\geqslant} : x_j \in [x_i]_A^{\wedge\geqslant}\}$;
(4) $[x_i]_A^{\wedge\geqslant} = [x_j]_A^{\wedge\geqslant}$ iff $f(x_i,a) = f(x_j,a)$ $(\forall a \in A)$.

Proof. Let $B \subseteq A \subseteq AT$, (1) and (2) are straightforward.

(3) If $x_j \in [x_i]_A^{\wedge\geqslant}$, it follows from the dominance relation $R_A^{\wedge\geqslant}$ that $f(x_i,a) \subseteq f(x_j,a)$ for arbitrary $a \in A$. Analogously, for $\forall x \in [x_j]_A^{\wedge\geqslant}$, we have $f(x_j,a) \subseteq f(x,a)$ for arbitrary $a \in A$. Hence, $f(x_i,a) \subseteq f(x,a)$ $(\forall a \in A)$. Thus we have $x \in [x_i]_A^{\wedge\geqslant}$, i.e., $[x_j]_A^{\wedge\geqslant} \subseteq [x_i]_A^{\wedge\geqslant}$. Therefore, $[x_i]_A^{\wedge\geqslant} = \bigcup\{[x_j]_A^{\wedge\geqslant} : x_j \in [x_i]_A^{\wedge\geqslant}\}$ holds.

(4) "$\Rightarrow$" When $[x_i]_A^{\wedge\geqslant} = [x_j]_A^{\wedge\geqslant}$, it follows from (3) that $[x_j]_A^{\wedge\geqslant} \subseteq [x_i]_A^{\wedge\geqslant}$, i.e., $f(x_i,a) \subseteq f(x_j,a)$ for arbitrary $a \in A$. Analogously, we have that $f(x_j,a) \subseteq f(x_i,a)$ for arbitrary $a \in A$. Hence, $f(x_i,a) = f(x_j,a)(\forall a \in A)$.

"$\Leftarrow$" If $f(x_i,a) = f(x_j,a)(\forall a \in A)$, from the definition of the set of objects dominating x, it is easy to get $[x_i]_A^{\wedge\geqslant} = [x_j]_A^{\wedge\geqslant}$.

This completes the proof. □

Let $U/R_A^{\wedge\geqslant}$ denote classification, which is the family set $\mathsf{F} = \{[x]_A^{\wedge\geqslant} | x \in U\}$. Any element from $U/R_A^{\wedge\geqslant}$ will be called a dominance class. All the dominance classes in $U/R_A^{\wedge\geqslant}$ do not constitute a partition of U in general. In fact, $\mathsf{F} = \{[x]_A^{\wedge\geqslant} | x \in U\}$ induces a covering of U, i.e., $\bigcup_{x\in U}[x]_A^{\wedge\geqslant} = U$. This is illustrated by the following example.

Example 3.2. From Table 2, one can obtain the following

$$U/R_{AT}^{\wedge\geqslant} = \{[x_1]_{AT}^{\wedge\geqslant}, [x_2]_{AT}^{\wedge\geqslant}, \ldots, [x_{10}]_{AT}^{\wedge\geqslant}\},$$

where $[x_1]^{\wedge\geqslant}_{AT} = \{x_1, x_2, x_3\}$, $[x_2]^{\wedge\geqslant}_{AT} = \{x_2\}$, $[x_3]^{\wedge\geqslant}_{AT} = \{x_2, x_3\}$, $[x_4]^{\wedge\geqslant}_{AT} = \{x_2, x_4, x_{10}\}$, $[x_5]^{\wedge\geqslant}_{AT} = \{x_2, x_5\}$, $[x_6]^{\wedge\geqslant}_{AT} = \{x_6\}$, $[x_7]^{\wedge\geqslant}_{AT} = \{x_7\}$, $[x_8]^{\wedge\geqslant}_{AT} = \{x_8\}$, $[x_9]^{\wedge\geqslant}_{AT} = \{x_2, x_9\}$ and $[x_{10}]^{\wedge\geqslant}_{AT} = \{x_2, x_{10}\}$. And

$$U/R^{\wedge\leqslant}_{AT} = \left\{[x_1]^{\wedge\leqslant}_{AT}, [x_2]^{\wedge\leqslant}_{AT}, \ldots, [x_{10}]^{\wedge\leqslant}_{AT}\right\},$$

where $[x_1]^{\wedge\leqslant}_{AT} = \{x_1\}$, $[x_2]^{\wedge\leqslant}_{AT} = \{x_1, x_2, x_3, x_4, x_5, x_9, x_{10}\}$, $[x_3]^{\wedge\leqslant}_{AT} = \{x_1, x_3\}$, $[x_4]^{\wedge\leqslant}_{AT} = \{x_4\}$, $[x_5]^{\wedge\leqslant}_{AT} = \{x_5\}$, $[x_6]^{\wedge\leqslant}_{AT} = \{x_6\}$, $[x_7]^{\wedge\leqslant}_{AT} = \{x_7\}$, $[x_8]^{\wedge\leqslant}_{AT} = \{x_8\}$, $[x_9]^{\wedge\leqslant}_{AT} = \{x_9\}$ and $[x_{10}]^{\wedge\leqslant}_{AT} = \{x_4, x_{10}\}$.

Obviously, one can note that $x_2 \succcurlyeq_{AT} x_3 \succcurlyeq_{AT} x_1$, $x_2 \succcurlyeq_{AT} x_{10} \succcurlyeq_{AT} x_4$, $x_2 \succcurlyeq_{AT} x_5$ and $x_2 \succcurlyeq_{AT} x_9$.

From the above denotations and properties, one can learn that in a set-valued information system, if the value field of each object is interpreted conjunctively, then the dominance relation $R^{\wedge\leqslant}_{AT}$ can be extracted from the set-valued information system.

3.2. Disjunctive set-valued ordered information systems

In this subsection, through an illustrative example, we examine disjunctive set-valued ordered information systems and obtain several their important properties. The following example presents a disjunctive set-valued information system.

Example 3.3. A set-valued information system is presented in Table 3, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ and $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

For a disjunctive set-valued information system $S = (U, AT, V, f)$, the relationships among any set $f(x, a)$, $x \in U$, $a \in AT$ are disjunctive. Hence, for convenience, let $R^{\vee\geqslant}_{A}$, $A \subseteq AT$, denote a binary dominance relation between objects that are possibly dominant in terms of values of attributes set A. Under this consideration, we call S a *disjunctive set-valued ordered information system*. Let us define the dominance relation more precisely as follows

$$R^{\vee\geqslant}_{A} = \{(y, x) \in U \times U | \forall a \in A, \exists u_y \in f(y, a), \exists v_x \in f(x, a) \text{ such that } u_y \geqslant v_x\}.$$

By the definition of the dominance relation $R^{\vee\geqslant}_{A}$, it can be observed that if a pair of objects (y, x) from $U \times U$ lies in $R^{\vee\geqslant}_{A}$, then they are perceived as y dominates x; in other words, y may have a better property than x with respect to A in reality. In fact, this dominance relation is equivalent to the representation given below

$$R^{\vee\geqslant}_{A} = \{(y, x) \in U \times U | \forall a \in A, \max f(y, a) \geqslant \min f(x, a)\}.$$

From the definition of $R^{\vee\geqslant}_{A}$, the following properties can be easily obtained.

Property 3.4. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued information system and $A \subseteq AT$, then*

$$R^{\vee\geqslant}_{A} = \bigcap_{a \in A} R^{\vee\geqslant}_{a}.$$

Property 3.5. *Let $R^{\vee\geqslant}_{A}$ be a dominance relation in a disjunctive set-valued information system, then the following properties hold*

(1) $R^{\vee\geqslant}_{A}$ is reflexive;
(2) $R^{\vee\geqslant}_{A}$ is non-symmetric;
(3) $R^{\vee\geqslant}_{A}$ is intransitive.

Furthermore, denoted by

$$[x]^{\vee\geqslant}_{A} = \{y \in U | (y, x) \in R^{\vee\geqslant}_{A}\},$$
$$[x]^{\vee\leqslant}_{A} = \{y \in U | (x, y) \in R^{\vee\geqslant}_{A}\},$$

where $[x]^{\vee\geqslant}_{A}$ describes objects that may dominate x and $[x]^{\vee\leqslant}_{A}$ describes objects that may be dominated by x in terms of A in a disjunctive set-valued information system.

Table 3
A set-valued information system.

U	a_1	a_2	a_3	a_4	a_5
x_1	{1}	{0, 1}	{0}	{1, 2}	{2}
x_2	{0, 1}	{2}	{1, 2}	{0}	{0}
x_3	{0}	{1, 2}	{1}	{0, 1}	{0}
x_4	{0}	{1}	{1}	{1}	{0, 2}
x_5	{2}	{1}	{0, 1}	{0}	{1}
x_6	{0, 2}	{1}	{0, 1}	{0}	{1}
x_7	{1}	{0, 2}	{0, 1}	{1}	{2}
x_8	{0}	{2}	{1}	{0}	{0, 1}
x_9	{1}	{0, 1}	{0, 2}	{1}	{2}
x_{10}	{1}	{1}	{2}	{0, 1}	{2}

Property 3.6. *Let $S=(U,AT,V,f)$ be a disjunctive set-valued ordered information system and $A,B\subseteq AT$, one has*

(1) if $B\subseteq A\subseteq AT$, then $R_B^{\vee\geqslant}\supseteq R_A^{\vee\geqslant}\supseteq R_{AT}^{\vee\geqslant}$;
(2) if $B\subseteq A\subseteq AT$, then $[x]_B^{\vee\geqslant}\supseteq [x]_A^{\vee\geqslant}\supseteq [x]_{AT}^{\vee\geqslant}$;
(3) if $\min f(x_i,a)=\min f(x_j,a)(\forall a\in A)$, then $[x_i]_A^{\vee\geqslant}=[x_j]_A^{\vee\geqslant}$.

Proof. Let $B\subseteq A\subseteq AT$, (1) and (2) are straightforward.

(3) For $\forall x\in U$, if $x\in[x_i]_A^{\vee\geqslant}$, it follows from the definition of the dominance relation $R_A^{\vee\geqslant}$ that $\max f(x,a)\geqslant\min f(x_i,a)$ $(\forall a\in A)$. Since the assumption $\min f(x_i,a)=\min f(x_j,a)(\forall a\in A)$, thus $\max f(x,a)\geqslant\min f(x_j,a)$ $(\forall a\in A)$, i.e., $x\in[x_j]_A^{\vee\geqslant}$. Hence, $[x_i]_A^{\vee\geqslant}\subseteq[x_j]_A^{\vee\geqslant}$ holds. Analogously, we can prove $[x_j]_A^{\vee\geqslant}\subseteq[x_i]_A^{\vee\geqslant}$. Therefore, *we have* $[x_i]_A^{\vee\geqslant}=[x_j]_A^{\vee\geqslant}$. This completes the proof. □

Let $U/R_A^{\vee\geqslant}$ denote classification, which is the family set $\mathsf{F}=\{[x]_A^{\vee\geqslant}|x\in U\}$. Any element from $U/R_A^{\vee\geqslant}$ will be called a dominance class. All the dominance classes in $U/R_A^{\vee\geqslant}$ do not constitute a partition of U in general. In fact, $\mathsf{F}=\{[x]_A^{\vee\geqslant}|x\in U\}$ induces a covering of U, i.e., $\bigcup_{x\in U}[x]_A^{\vee\geqslant}=U$. This is illustrated in Example 3.4.

Example 3.4. From Table 3, one has

$$U/R_{AT}^{\vee\geqslant}=\left\{[x_1]_{AT}^{\vee\geqslant},[x_2]_{AT}^{\vee\geqslant},\ldots,[x_{10}]_{AT}^{\vee\geqslant}\right\},$$

where $[x_1]_{AT}^{\vee\geqslant}=[x_7]_{AT}^{\vee\geqslant}=[x_9]_{AT}^{\vee\geqslant}=\{x_1,x_7,x_9,x_{10}\}$, $[x_2]_{AT}^{\vee\geqslant}=[x_8]_{AT}^{\vee\geqslant}=\{x_2,x_3,x_7,x_8\}$, $[x_3]_{AT}^{\vee\geqslant}=\{x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}\}$, $[x_4]_{AT}^{\vee\geqslant}=\{x_3,x_4,x_7,x_8,x_9,x_{10}\}$, $[x_5]_{AT}^{\vee\geqslant}=\{x_5,x_6\}$, $[x_6]_{AT}^{\vee\geqslant}=\{x_1,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}\}$ $[x_{10}]_{AT}^{\vee\geqslant}=\{x_9,x_{10}\}$.

From Example 3.4, one can easily notice that unlike an OIS, the property "if $x_j\in[x_i]_A^{\vee\geqslant}$, then $[x_j]_A^{\vee\geqslant}\subseteq[x_i]_A^{\vee\geqslant}$ and $[x_i]_A^{\vee\geqslant}=\bigcup\{[x_j]_A^{\vee\geqslant}:x_j\in[x_i]_A^{\vee\geqslant}\}$" does not hold.

Similarly, one also can see that if the value field of each object is interpreted disjunctively in a set-valued information system, then the dominance relation $R_{AT}^{\vee\leqslant}$ can be extracted from the set-valued information system.

3.3. Ranking for all objects in set-valued ordered information systems

First, we present how to rank all objects in a conjunctive set-valued ordered information system. Unlike some existing ordered information systems, it is very difficult to give the ordering of all objects through using the dominance relation $R_A^{\wedge\geqslant}$ in conjunctive set-valued ordered information systems in general. In particular, when there is an inclusion relation between the values of any two objects, i.e., $f(x_i,a)\subseteq f(x_j,a)$ or $f(x_i,a)\supseteq f(x_j,a)$ $(i,j\in U,a\in A)$, this type of information systems will have the same ordering properties as classical ordered information systems. In this situation, through using the dominance relation $R_A^{\wedge\geqslant}$, one gives the ordering of all objects. We call the type of information systems *conjunctive set-valued whole ordered information systems*.

Definition 3.2. Let $S=(U,AT,V,f)$ be a conjunctive set-valued whole OIS and $A\subseteq AT$. *Dominance degree* between two objects with respect to A is defined as

$$D_A(x_i,x_j)=\frac{|\sim[x_i]_A^{\wedge\geqslant}\cup[x_j]_A^{\wedge\geqslant}|}{|U|},$$

where $|\cdot|$ denotes the cardinality of a set and $x_i,x_j\in U$.

Obviously, $\frac{1}{|U|}\leqslant D_A(x_i,x_j)\leqslant 1$. From Definition 3.2, one can construct a *dominance relation matrix* with respect to A. From this matrix, the dominance degree of each object can be calculated according to the following formula

$$D_A(x_i)=\frac{1}{|U|-1}\sum_{j\neq i}D_A(x_i,x_j),\quad x_i,x_j\in U.$$

Given the dominance degree of each object on the universe, one can rank all objects according to the number of $D_A(x_i)$. Higher value implies more suitable object.

Example 3.5. Table 4 is a conjunctive set-valued whole OIS. Rank all objects in U according to the dominance relation $R_{AT}^{\wedge\geqslant}$. By computing, we have

$$[x_1]_{AT}^{\wedge\geqslant}=\{x_1,x_3\},\quad [x_2]_{AT}^{\wedge\geqslant}=\{x_1,x_2,x_3\},\quad [x_3]_{AT}^{\wedge\geqslant}=\{x_3\},\quad [x_4]_{AT}^{\wedge\geqslant}=\{x_3,x_4\},$$
$$[x_5]_{AT}^{\wedge\geqslant}=\{x_1,x_2,x_3,x_4,x_5,\quad x_6\},[x_6]_{AT}^{\wedge\geqslant}=\{x_1,x_3,x_6\}.$$

Hence, from Definition 3.2, we can get the following dominance relation

$$D_{AT}=\begin{pmatrix}1&1&0.83&0.83&1&1\\0.83&1&0.67&0.67&1&0.83\\1&1&1&1&1&1\\0.83&0.83&0.83&1&1&0.83\\0.33&0.50&0.17&0.33&1&0.50\\0.83&0.83&0.67&0.67&1&1\end{pmatrix}.$$

Table 4
A "$\bigwedge$" set-valued whole ordered information system.

U	a_1	a_2	a_3
x_1	$\{E,F,G\}$	$\{F,G\}$	$\{E,G\}$
x_2	$\{E,F\}$	$\{F,G\}$	$\{E,G\}$
x_3	$\{E,F,G\}$	$\{E,F,G\}$	$\{E,F,G\}$
x_4	$\{E,F\}$	$\{F\}$	$\{E,F,G\}$
x_5	$\{E\}$	$\{F\}$	$\{G\}$
x_6	$\{E,F,G\}$	$\{F,G\}$	$\{G\}$

Therefore, we have

$$D_{AT}(x_1) = 0.93, \quad D_{AT}(x_2) = 0.80, \quad D_{AT}(x_3) = 1.00, \quad D_{AT}(x_4) = 0.86, \quad D_{AT}(x_5) = 0.37, \quad D_{AT}(x_6) = 0.80.$$

In the following, ranking objects according to the number of $D_{AT}(x_i)$, a object with larger number may imply a more suitable object.

$$x_3 \succcurlyeq x_1 \succcurlyeq x_4 \succcurlyeq \begin{pmatrix} x_2 \\ x_6 \end{pmatrix} \succcurlyeq x_5.$$

Then, we discuss how to make a decision in a disjunctive set-valued ordered information system.

Definition 3.3. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$, the *dominance degree* between two objects with respect to A is defined as

$$D_A(x_i, x_j) = \frac{|\sim [x_i]_A^{\vee\geqslant} \cup [x_j]_A^{\vee\geqslant}|}{|U|},$$

where $|\cdot|$ denotes the cardinality of a set, $x_i, x_j \in U$.

Obviously, $\frac{1}{|U|} \leqslant D_A(x_i, x_j) \leqslant 1$. From Definition 3.3, we can construct a *dominance relation matrix* with respect to A. From this matrix, the dominance degree of each object can be calculated as follows

$$D_A(x_i) = \frac{1}{|U| - 1} \sum_{j \neq i} D_A(x_i, x_j), \quad x_i, x_j \in U.$$

Next we can rank all objects according to the number of $D_A(x_i)$.

Example 3.6. Continuing Example 3.4, let us rank objects in U according to the dominance relation $R_{AT}^{\vee\geqslant}$. The obtained dominance relation comes with the entries

$$D_{AT} = \begin{pmatrix} 1 & 0.7 & 0.9 & 0.9 & 0.6 & 1 & 1 & 0.7 & 1 & 0.8 \\ 0.7 & 1 & 1 & 0.9 & 0.6 & 0.8 & 0.7 & 1 & 0.7 & 0.6 \\ 0.4 & 0.5 & 1 & 0.7 & 0.3 & 0.8 & 0.4 & 0.5 & 0.4 & 0.3 \\ 0.7 & 0.7 & 1 & 1 & 0.4 & 0.9 & 0.7 & 0.7 & 0.7 & 0.6 \\ 0.8 & 0.8 & 1 & 0.8 & 1 & 1 & 0.8 & 0.8 & 0.8 & 0.8 \\ 0.6 & 0.4 & 0.9 & 0.7 & 0.4 & 1 & 0.6 & 0.4 & 0.6 & 0.4 \\ 1 & 0.7 & 0.9 & 0.9 & 0.6 & 1 & 1 & 0.7 & 1 & 0.8 \\ 0.7 & 1 & 1 & 0.9 & 0.6 & 0.8 & 0.7 & 1 & 0.7 & 0.6 \\ 1 & 0.7 & 0.9 & 0.9 & 0.6 & 1 & 1 & 0.7 & 1 & 0.8 \\ 1 & 0.8 & 1 & 1 & 0.8 & 1 & 1 & 0.8 & 1 & 1 \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned} &D_{AT}(x_1) = 0.84, \quad D_{AT}(x_2) = 0.78, \quad D_{AT}(x_3) = 0.48, \quad D_{AT}(x_4) = 0.71, \\ &D_{AT}(x_5) = 0.84, \quad D_{AT}(x_6) = 0.56, \quad D_{AT}(x_7) = 0.84, \quad D_{AT}(x_8) = 0.78, \\ &D_{AT}(x_9) = 0.84, \quad D_{AT}(x_{10}) = 0.93. \end{aligned}$$

The ranking produced in this way comes in the form

$$x_{10} \succcurlyeq \begin{pmatrix} x_1 \\ x_5 \\ x_7 \\ x_9 \end{pmatrix} \succcurlyeq \begin{pmatrix} x_2 \\ x_8 \end{pmatrix} \succcurlyeq x_4 \succcurlyeq x_6 \succcurlyeq x_3.$$

3.4. Rough set approach to set-valued ordered information systems

In this subsection, a rough set approach to set-valued ordered information systems will be introduced and its several important properties are investigated.

The following definition deals with the approximations of a set in a conjunctive set-valued OIS.

Definition 3.4. Let $S = (U, AT, V, f)$ be a conjunctive set-valued OIS. For any $X \subseteq U$ and $A \subseteq AT$, the lower and upper approximations of X with respect to the dominance relation $R_A^{\wedge\geqslant}$ are defined as follows

$$\underline{R_A^{\wedge\geqslant}}(X) = \{x \in U | [x]_A^{\wedge\geqslant} \subseteq X\},$$
$$\overline{R_A^{\wedge\geqslant}}(X) = \{x \in U | [x]_A^{\wedge\geqslant} \cap X \neq \emptyset\}.$$

From Definition 3.4, one can easily notice that $\underline{R_A^{\wedge\geqslant}(X)}$ is a set of objects that belong to X with certainty, whereas $\overline{R_A^{\wedge\geqslant}}(X)$ is a set of objects that possibly belong to X. $Bn_A(X) = \overline{R_A^{\wedge\geqslant}}(X) - \underline{R_A^{\wedge\geqslant}}(X)$ denotes the boundary of the rough set.

Example 3.7. We continue Example 3.2. Let $X = \{x_2, x_5, x_6\}$, then one has

$$\underline{R_A^{\wedge\geqslant}}(X) = \{x_2, x_5, x_6\} \subseteq X, \quad \overline{R_A^{\wedge\geqslant}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}\} \supseteq X.$$

From Definition 3.4, the following properties are derived.

Property 3.7. *Let $S = (U, AT, V, f)$ be a conjunctive set-valued OIS, $X \subseteq U$ and $A \subseteq AT$, then*

(1) $\underline{R_A^{\wedge\geqslant}}(\emptyset) = \overline{R_A^{\wedge\geqslant}}(\emptyset) = \emptyset, \underline{R_A^{\wedge\geqslant}}(U) = \overline{R_A^{\wedge\geqslant}}(U) = U$;
(2) $\underline{R_A^{\wedge\geqslant}}(X) \subseteq X \subseteq \overline{R_A^{\wedge\geqslant}}(X)$;
(3) $\underline{R_A^{\wedge\geqslant}}(\underline{R_A^{\wedge\geqslant}}(X)) = \underline{R_A^{\wedge\geqslant}}(X), \overline{R_A^{\wedge\geqslant}}(\overline{R_A^{\wedge\geqslant}}(X)) = \overline{R_A^{\wedge\geqslant}}(X)$;
(4) $\underline{R_A^{\wedge\geqslant}}(X) = \sim \overline{R_A^{\wedge\geqslant}}(\sim X), \overline{R_A^{\wedge\geqslant}}(X) = \sim \underline{R_A^{\wedge\geqslant}}(\sim X)$;
(5) $\underline{R_A^{\wedge\geqslant}}(X) \subseteq \underline{R_{AT}^{\wedge\geqslant}}(X), \overline{R_A^{\wedge\geqslant}}(X) \supseteq \overline{R_{AT}^{\wedge\geqslant}}(X), Bn_{AT}(X) \subseteq Bn_A(X)$.

Proof. This proof is similar to that of Property 4.1 in [38] and that of Property 2 in [40]. □

Property 3.8. *Let $S = (U, AT, V, f)$ be a conjunctive set-valued OIS, $X, Y \subseteq U$ and $A \subseteq AT$, then*

(1) *if* $X \subseteq Y$, *then* $\underline{R_A^{\wedge\geqslant}}(X) \subseteq \underline{R_A^{\wedge\geqslant}}(Y)$, $\overline{R_A^{\wedge\geqslant}}(X) \subseteq \overline{R_A^{\wedge\geqslant}}(Y)$;
(2) $\underline{R_A^{\wedge\geqslant}}(X \cap Y) = \underline{R_A^{\wedge\geqslant}}(X) \cap \underline{R_A^{\wedge\geqslant}}(Y)$;
(3) $\overline{R_A^{\wedge\geqslant}}(X \cup Y) = \overline{R_A^{\wedge\geqslant}}(X) \cup \overline{R_A^{\wedge\geqslant}}(Y)$;
(4) $\overline{R_A^{\wedge\geqslant}}(X \cap Y) \subseteq \overline{R_A^{\wedge\geqslant}}(X) \cap \overline{R_A^{\wedge\geqslant}}(Y)$;
(5) $\underline{R_A^{\wedge\geqslant}}(X \cup Y) \supseteq \underline{R_A^{\wedge\geqslant}}(X) \cup \underline{R_A^{\wedge\geqslant}}(Y)$.

Proof. This proof is similar to that of Property 4.2 in [38] and that of Property 2 in [40]. □

The lower and upper approximations of X expressed with respect to the dominance relation $R_A^{\wedge\geqslant}$ can be used to extract decision rules, where $\underline{R_A^{\wedge\geqslant}}(X)$ can extract decision rules with certainty, while $Bn_A(X) = \overline{R_A^{\wedge\geqslant}}(X) - \underline{R_A^{\wedge\geqslant}}(X)$ can extract possible decision rules.

Now we investigate the problem of set approximation in disjunctive set-valued ordered information systems.

Definition 3.5. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS. For any $X \subseteq U$ and $A \subseteq AT$, the lower and upper approximations of X with respect to the dominance relation $R_A^{\vee\geqslant}$ are defined as follows

$$\underline{R_A^{\vee\geqslant}}(X) = \{x \in U | [x]_A^{\vee\geqslant} \subseteq X\},$$
$$\overline{R_A^{\vee\geqslant}}(X) = \{x \in U | [x]_A^{\vee\geqslant} \cap X \neq \emptyset\}.$$

From Definition 3.5, one can easily notice that $\underline{R_A^{\vee\geqslant}(X)}$ is a set of objects that belong to X with certainty, whereas $\overline{R_A^{\vee\geqslant}}(X)$ is a set of objects that possibly belong to X. $Bn_A(X) = \overline{R_A^{\vee\geqslant}}(X) - \underline{R_A^{\vee\geqslant}}(X)$ denotes the boundary of the rough set.

Example 3.8. This is a continuation of Example 3.4.

Let $X = \{x_2, x_5, x_6\}$, then we have

$$\underline{R_A^{\vee\geqslant}}(X) = \{x_5\} \subseteq X, \quad \overline{R_A^{\vee\geqslant}}(X) = \{x_2, x_3, x_5, x_6, x_8\} \supseteq X.$$

From Definition 3.5, one can easily obtain the following properties.

Property 3.9. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $X \subseteq U$ and $A \subseteq AT$, then*

(1) $\underline{R_A^{\vee\geqslant}}(\emptyset) = \overline{R_A^{\vee\geqslant}}(\emptyset) = \emptyset$, $\underline{R_A^{\vee\geqslant}}(U) = \overline{R_A^{\vee\geqslant}}(U) = U$;
(2) $\underline{R_A^{\vee\geqslant}}(X) \subseteq X \subseteq \overline{R_A^{\vee\geqslant}}(X)$;
(3) $\underline{R_A^{\vee\geqslant}}(\underline{R_A^{\vee\geqslant}}(X)) = \underline{R_A^{\vee\geqslant}}(X)$, $\overline{R_A^{\vee\geqslant}}(\overline{R_A^{\vee\geqslant}}(X)) = \overline{R_A^{\vee\geqslant}}(X)$;
(4) $\underline{R_A^{\vee\geqslant}}(X) = \sim \overline{R_A^{\vee\geqslant}}(\sim X)$, $\overline{R_A^{\vee\geqslant}}(X) = \sim \underline{R_A^{\vee\geqslant}}(\sim X)$;
(5) $\underline{R_A^{\vee\geqslant}}(X) \subseteq \underline{R_{AT}^{\vee\geqslant}}(X)$, $\overline{R_A^{\vee\geqslant}}(X) \supseteq \overline{R_{AT}^{\vee\geqslant}}(X)$, $Bn_{AT}(X) \subseteq Bn_A(X)$.

Proof. This proof is similar to that of Property 4.1 in [38] and that of Property 2 in [40]. □

Property 3.10. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $X, Y \subseteq U$ and $A \subseteq AT$, then*

(1) if $X \subseteq Y$, then $\underline{R_A^{\vee\geqslant}}(X) \subseteq \underline{R_A^{\vee\geqslant}}(Y)$, $\overline{R_A^{\vee\geqslant}}(X) \subseteq \overline{R_A^{\vee\geqslant}}(Y)$;
(2) $\underline{R_A^{\vee\geqslant}}(X \cap Y) = \underline{R_A^{\vee\geqslant}}(X) \cap \underline{R_A^{\vee\geqslant}}(Y)$;
(3) $\overline{R_A^{\vee\geqslant}}(X \cup Y) = \overline{R_A^{\vee\geqslant}}(X) \cup \overline{R_A^{\vee\geqslant}}(Y)$;
(4) $\overline{R_A^{\vee\geqslant}}(X \cap Y) \subseteq \overline{R_A^{\vee\geqslant}}(X) \cap \overline{R_A^{\vee\geqslant}}(Y)$;
(5) $\underline{R_A^{\vee\geqslant}}(X \cup Y) \supseteq \underline{R_A^{\vee\geqslant}}(X) \cup \underline{R_A^{\vee\geqslant}}(Y)$.

Proof. Refer to [38,40]. □

The lower and upper approximations of X completed with respect to the dominance relation $R_A^{\vee\geqslant}$ can also be used to extract decision rules, where $\underline{R_A^{\vee\geqslant}}(X)$ can extract decision rules with certainty, while $Bn_A(X) = \overline{R_A^{\vee\geqslant}}(X) - \underline{R_A^{\vee\geqslant}}(X)$ can extract possible decision rules.

4. Set-valued ordered decision tables and their decision rules

In this section, we investigate two types of set-valued ordered decision tables and decision rules from these two types of decision tables.

4.1. Set-valued ordered decision tables

A *set-valued ordered decision table* (ODT) is a set-valued ordered information system $S = (U, C \cup d, V, f)$, where $d(d \notin C$ and $f(x, d)(x \in U)$ is single-valued) is an overall preference called the decision, and all the elements of C are criteria, and $f : U \times C \to 2^V$ is a set-valued mapping.

Furthermore, let us assume that the decision attribute d makes a partition of U into a finite number of classes; let $\mathbf{D} = \{D_1, D_2, \ldots, D_r\}$ be a set of these classes that are ordered, that is, for all $i, j \leqslant r$ if $i \geqslant j$, then the objects from D_i are preferred to the objects from D_j.

The sets to be approximated are an upward union and a downward union of classes, which are defined as follows

$$D_i^{\geqslant} = \bigcup_{j \leqslant i} D_j, \quad D_i^{\leqslant} = \bigcup_{j \geqslant i} D_j, \quad (i \leqslant r).$$

The statement $x \in D_i^{\geqslant}$ means "x belongs to at least class D_i", whereas $x \in D_i^{\leqslant}$ means "x belongs to at most class D_i".

Analogous to the idea of decision approximation in [36], in the following, we give the definitions of the lower and upper approximations of $D_i^{\geqslant}(i \leqslant r)$ with respect to the dominance relation $R_A^{\Delta\geqslant}$ $(\Delta = \wedge, \vee)$ in a set-valued ODT.

Definition 4.1. Let $S = (U, C \cup d, V, f)$ be a set-valued ODT, $A \subseteq C$ and $\mathbf{D} = \{D_1, D_2, \ldots, D_r\}$ is the decision induced by d, the lower and upper approximations of $D_i^{\geqslant}(i \leqslant r)$ with respect to the dominance relation $R_A^{\Delta\geqslant}$ $(\Delta = \wedge, \vee)$ are defined as

$$\underline{R_A^{\Delta\geqslant}}(D_i^{\geqslant}) = \{x \in U | [x]_A^{\Delta\geqslant} \subseteq D_i^{\geqslant}\},$$
$$\overline{R_A^{\Delta\geqslant}}(D_i^{\geqslant}) = \bigcup_{x \in D_i^{\geqslant}} [x]_A^{\Delta\geqslant}.$$

Similarly, we define the lower and upper approximations of $D_i^{\leqslant}$ $(i \leqslant r)$ with respect to the dominance relation $R_A^{\Delta\geqslant}$ $(\Delta = \wedge, \vee)$ in a set-valued ODT.

Definition 4.2. Let $S = (U, C \cup d, V, f)$ be a set-valued ODT, $A \subseteq C$ and $\mathbf{D} = \{D_1, D_2, \ldots, D_r\}$ is the decision induced by d, the lower and upper approximations of $D_i^{\leqslant}(i \leqslant r)$ with respect to the dominance relation $R_A^{\Delta\geqslant}$ $(\Delta = \wedge, \vee)$ are defined as

$$\underline{R_A^{\Delta\geqslant}}(D_i^{\leqslant}) = \{x \in U | [x]_A^{\Delta\leqslant} \subseteq D_i^{\leqslant}\},$$
$$\overline{R_A^{\Delta\geqslant}}(D_i^{\leqslant}) = \bigcup_{x \in D_i^{\leqslant}} [x]_A^{\Delta\leqslant}.$$

Naturally, the A-boundaries of $D_i^{\geqslant}$ $(i \leqslant r)$ and $D_i^{\leqslant}$ $(i \leqslant r)$ can be defined as

$$Bn_A(D_i^{\geqslant}) = \overline{R_A^{\Delta\geqslant}}(D_i^{\geqslant}) - \underline{R_A^{\Delta\geqslant}}(D_i^{\geqslant}),$$
$$Bn_A(D_i^{\leqslant}) = \overline{R_A^{\Delta\geqslant}}(D_i^{\leqslant}) - \underline{R_A^{\Delta\geqslant}}(D_i^{\leqslant}).$$

The lower approximations $\underline{R_A^{\Delta\geqslant}}(D_i^{\geqslant})$ and $\underline{R_A^{\Delta\geqslant}}(D_i^{\leqslant})$ can be used to extract certain decision rules, while the boundaries $Bn_A(D_i^{\geqslant})$ and $Bn_A(D_i^{\leqslant})$ can be used to induce possible decision rules from a set-valued ordered decision table.

4.2. Decision rules from conjunctive set-valued ordered decision tables

In [39], an atomic expression over a single attribute a is defined as either $(a, \geqslant)$ (according to increasing preference) or $(a, \leqslant)$ (according to decreasing preference) in an ordered information system. For any $A \subseteq AT$, an expression over A in ordered information systems is defined by $\bigwedge_{a \in A} e(a)$, where $e(a)$ is an atomic expression over a. The set of all expression over A in an OIS is denoted by $E(A)$. For instance, in Table 1, $AT = \{a_1, a_2, a_3\}$, the set of $E(AT)$ is

$$E(\{a_1, a_2, a_3\}) = \{(a_1, \geqslant) \wedge (a_2, \geqslant) \wedge (a_3, \geqslant), (a_1, \geqslant) \wedge (a_2, \geqslant) \wedge (a_3, \leqslant), \ldots, (a_1, \leqslant) \wedge (a_2, \leqslant) \wedge (a_3, \leqslant)\}.$$

In an OIS, $a \in AT$, $v_1 \in V_a$, an atomic formula over a single attribute a ia defined as either $(a, \geqslant, v_1)$ (according to increasing preference) or $(a, \leqslant, v_1)$ (according to decreasing preference). For any $A \subseteq AT$, a formula over A in OIS is defined by $\bigwedge_{a \in A} m(a)$, where $m(a)$ is an atomic formula over a. The set of all formulas over A in an OIS is denoted by $M(A)$. Let the formula $\phi \in M(A)$, $\|\phi\|$ denotes the set of objects satisfying formula ϕ. For example, $(a, \geqslant, v_1)$ and $(a, \leqslant, v_1)$ are atomic formulas, then

$$\|(a, \geqslant, v_1)\| = \{x \in U | f(x, a) \geqslant v_1\},$$
$$\|(a, \leqslant, v_1)\| = \{x \in U | f(x, a) \leqslant v_1\}.$$

However, in a conjunctive set-valued ordered information system, the mapping $f : U \times A \to V$ is not single-valued but set-valued. Hence, we modify the definition of a formula over a according to the dominance relation $R_A^{\wedge\geqslant}$

$$\|(a, \supseteq, v_1)\| = \{x \in U | f(x, a) \supseteq v_1\},$$
$$\|(a, \subseteq, v_1)\| = \{x \in U | f(x, a) \subseteq v_1\}.$$

Now we consider a conjunctive set-valued ODT $S = (U, C \cup \{d\}, V, f)$, a subset of attributes $A \subseteq C$. For two formulas $\phi \in M(A)$ and $\varphi \in M(d)$, a decision rule, denoted by $\phi \to \varphi$, is read "if ϕ then φ". The formula ϕ is called the rule's *antecedent*, and the formula φ is called the rule's *consequent*. We say that an object supports a decision rule if it matches both the condition and the decision parts of the rule. On the other hand, an object is covered by a decision rule if it matches the condition parts of the rule. A decision rule states how "evaluation of objects on attributes A is at least as good as a given level" or "evaluation of objects on attributes A is at most as good as a given level" determines "objects belong (or possibly belong) to at least a given class" or "objects belong (or possibly belong) to at most a given class."

Like decision rules shown in [40], there are four types of decision rules to be considered

(1) certain $\supseteq$-decision rules with the following syntax:
if $(f(x, a_1) \supseteq v_{a_1}) \wedge (f(x, a_2) \supseteq v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \supseteq v_{a_k}) \wedge (f(x, a_{k+1}) \subseteq v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \subseteq v_{a_p})$, then $x \in D_i^{\geqslant}$;
(2) possible $\supseteq$-decision rules with the following syntax:
if $(f(x, a_1) \supseteq v_{a_1}) \wedge (f(x, a_2) \supseteq v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \supseteq v_{a_k}) \wedge (f(x, a_{k+1}) \subseteq v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \subseteq v_{a_p})$, then x could belong to $D_i^{\geqslant}$;
(3) certain $\subseteq$-decision rules with the following syntax:
if $(f(x, a_1) \subseteq v_{a_1}) \wedge (f(x, a_2) \subseteq v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \subseteq v_{a_k}) \wedge (f(x, a_{k+1}) \supseteq v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \supseteq v_{a_p})$, then $x \in D_i^{\leqslant}$;
(4) possible $\subseteq$-decision rules with the following syntax:
if $(f(x, a_1) \subseteq v_{a_1}) \wedge (f(x, a_2) \subseteq v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \subseteq v_{a_k}) \wedge (f(x, a_{k+1}) \supseteq v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \supseteq v_{a_p})$, then x could belong to $D_i^{\leqslant}$;

where $O_1 = \{a_1, a_2, \ldots, a_k\} \subseteq C$, $O_2 = \{a_{k+1}, a_{k+2}, \ldots, a_p\} \subseteq C$, $C = O_1 \cup O_2$, O_1 with inclusion increasing preference and O_2 with inclusion decreasing preference, $(v_{a_1}, v_{a_2}, \ldots, v_{a_p}) \in V_{a_1} \times V_{a_2} \times \cdots \times V_{a_p}$, $i \leqslant r$.

Therefore, in a "$\bigwedge$" set-valued ODT, for a given upward or downward union $D_i^{\geqslant}$ or $D_j^{\leqslant}$, $i, j \leqslant r$, the rules induced under a hypothesis that objects belonging to $\underline{R_A^{\wedge\geqslant}}(D_i^{\geqslant})$ or to $\underline{R_A^{\wedge\leqslant}}(D_i^{\leqslant})$ are positive and all the others negative suggest the assignment of an object to "at least class D_i" or to "at most class D_j", respectively. Similarly, the rules induced under a hypothesis that objects belonging to $\overline{R_A^{\wedge\geqslant}}(D_i^{\geqslant})$ or to $\overline{R_A^{\wedge\leqslant}}(D_i^{\leqslant})$ are positive and all the others negative suggest the assignment of an object could belongs to "at least class D_i" or to "at most class D_j", respectively.

Now we employ an example to illustrate conjunctive set-valued ODT and decision rules extracted from this type of ODT in the following.

Example 4.1. Let us consider a conjunctive set-valued ODT, constructed from a conjunctive set-valued OIS in Table 2 and extended by decision attributes d as shown in Table 5.

From Table 5, it is easy to see that $\mathbf{D} = \{D_1, D_2\}$, where

$$D_1 = \{x_2, x_3, x_7, x_8, x_{10}\}, \quad D_2 = \{x_1, x_4, x_5, x_6, x_9\}.$$

In this conjunctive ordered decision table, because only two decision classes are considered, we have $D_1^{\geqslant} = D_1$ and $D_2^{\leqslant} = D_2$.

From Definition 4.1, we have

$$\underline{R_C^{\wedge\geqslant}}(D_1^{\geqslant}) = \{x_2, x_3, x_7, x_8, x_{10}\},$$

$$\overline{R_C^{\wedge\geqslant}}(D_1^{\geqslant}) = \{x_2, x_3, x_7, x_8, x_{10}\},$$

$$Bn_C(D_1^{\geqslant}) = \emptyset.$$

And, it is easily follows from Definition 4.2 that

$$\underline{R_C^{\wedge\geqslant}}(D_2^{\leqslant}) = \{x_1, x_4, x_5, x_6, x_9\},$$

$$\overline{R_C^{\wedge\geqslant}}(D_2^{\leqslant}) = \{x_1, x_4, x_5, x_6, x_9\},$$

$$Bn_C(D_2^{\leqslant}) = \emptyset.$$

One can obtain the following set of decision rules from the considered conjunctive ordered decision table:

r_1 : (Audition, $\supseteq$, $\{E,G\}$) $\wedge$ (Spoken language, $\supseteq$, $\{E,F\}$) $\wedge$ (Reading, $\supseteq$, $\{F,G\}$) $\wedge$ (Writing, $\supseteq$, $\{F,G\}$) $\to$ (d, $\geqslant$, *Good*)//supported by objects x_2, x_3;
r_2 : (Audition, $\supseteq$, $\{E,F,G\}$) $\wedge$ (Spoken language, $\supseteq$, $\{E,F,G\}$) $\wedge$ (Reading, $\supseteq$, $\{E,G\}$) $\wedge$ (Writing, $\supseteq$, $\{E,F,G\}$) $\to$ (d, $\geqslant$, *Good*)// supported by objects x_7;
r_3 : (Audition, $\supseteq$, $\{E,F\}$) $\wedge$ (Spoken language, $\supseteq$, $\{F,G\}$) $\wedge$ (Reading, $\supseteq$, $\{E,F,G\}$) $\wedge$ (Writing, $\supseteq$, $\{E,G\}$) $\to$ (d, $\geqslant$, *Good*)//supported by objects x_8;
r_4 : (Audition, $\supseteq$, $\{E,F\}$) $\wedge$ (Spoken language, $\supseteq$, $\{E,G\}$) $\wedge$ (Reading, $\supseteq$, $\{F,G\}$) $\wedge$ (Writing, $\supseteq$, $\{E,F\}$) $\to$ (d, $\geqslant$, *Good*)//supported by objects x_2,x_{10};
r_5 : (Audition, $\subseteq$, $\{E\}$) $\wedge$ (Spoken language, $\subseteq$, $\{E\}$) $\wedge$ (Reading, $\subseteq$, $\{F,G\}$) $\wedge$ (Writing, $\subseteq$, $\{F,G\}$) $\to$ (d, $\leqslant$, *Poor*)//supported by objects x_1;
r_6 : (Audition, $\subseteq$, $\{E,F\}$) $\wedge$ (Spoken language, $\subseteq$, $\{E,G\}$) $\wedge$ (Reading, $\subseteq$, $\{F,G\}$) $\wedge$ (Writing, $\subseteq$, $\{F\}$) $\to$ (d, $\leqslant$, *Poor*)//supported by objects x_4;
r_7 : (Audition, $\subseteq$, $\{F,G\}$) $\wedge$ (Spoken language, $\subseteq$, $\{F,G\}$) $\wedge$ (Reading, $\subseteq$, $\{F,G\}$) $\wedge$ (Writing, $\subseteq$, $\{F\}$) $\to$ (d, $\leqslant$, *Poor*)//supported by objects x_5;
r_8 : (Audition, $\subseteq$, $\{F\}$) $\wedge$ (Spoken language, $\subseteq$, $\{F\}$) $\wedge$ (Reading, $\subseteq$, $\{E,F\}$) $\wedge$ (Writing, $\subseteq$, $\{E,F\}$) $\to$ (d, $\leqslant$, *Poor*)//supported by objects x_6;
r_9 : (Audition, $\subseteq$, $\{F,G\}$) $\wedge$ (Spoken language, $\subseteq$, $\{G\}$) $\wedge$ (Reading, $\subseteq$, $\{F,G\}$) $\wedge$ (*Writing*, $\subseteq$, $\{F,G\}$) $\to$ (d, $\leqslant$, *Poor*)//supported by objects x_9.

where r_1, r_2, r_3, r_4 are certain $\supseteq$-decision rules, r_5, r_6, r_7, r_8, r_9 are certain $\subseteq$-decision rules.

For any ordering decision rule $r : \phi \to \varphi$, the certainty factor, support factor and coverage factor can be defined as follows

Table 5
A "$\bigwedge$" set-valued ordered decision table about language ability.

U	Audition	Spoken language	Reading	Writing	d
x_1	$\{E\}$	$\{E\}$	$\{F,G\}$	$\{F,G\}$	*Poor*
x_2	$\{E,F,G\}$	$\{E,F,G\}$	$\{F,G\}$	$\{E,F,G\}$	*Good*
x_3	$\{E,G\}$	$\{E,F\}$	$\{F,G\}$	$\{F,G\}$	*Good*
x_4	$\{E,F\}$	$\{E,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_5	$\{F,G\}$	$\{F,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_6	$\{F\}$	$\{F\}$	$\{E,F\}$	$\{E,F\}$	*Poor*
x_7	$\{E,F,G\}$	$\{E,F,G\}$	$\{E,G\}$	$\{E,F,G\}$	*Good*
x_8	$\{E,F\}$	$\{F,G\}$	$\{E,F,G\}$	$\{E,G\}$	*Good*
x_9	$\{F,G\}$	$\{G\}$	$\{F,G\}$	$\{F,G\}$	*Poor*
x_{10}	$\{E,F\}$	$\{E,G\}$	$\{F,G\}$	$\{E,F\}$	*Good*

$$cer(\phi \to \varphi) = \frac{card(\|\phi \wedge \varphi\|)}{card(\|\phi\|)},$$
$$sup(\phi \to \varphi) = \frac{card(\|\phi \wedge \varphi\|)}{card(|U|)},$$

and

$$cov(\phi \to \varphi) = \frac{card(\|\phi \wedge \varphi\|)}{card(\|\varphi\|)}.$$

The certainty factor can be interpreted as the frequency of objects having the property φ in the set of objects having the property ϕ and the coverage factor as the frequency of objects having the property ϕ in the set of objects having the property φ. While the support factor denotes the probability of objects having both the property ϕ and the property φ within the universe U.

Example 4.2. Compute three factors of the decision rule r_1 in Example 4.1.

r_1 : (Audition, $\supseteq, \{E, G\}) \wedge$ (Spoken language, $\supseteq, \{E, F\}) \wedge$ (Reading, $\supseteq, \{F, G\}) \wedge$ (Writing, $\supseteq, \{F, G\}) \to (d, \geqslant, Good)$.

We have $card(\|\phi\|) = 2$, $card(\|\phi \wedge \varphi\|) = 2$, $card(\|\varphi\|) = 5$ and $|U| = 10$; hence $cer(r_1) = 1$, $sup(r_1) = 0.2$ and $cov(r_1) = 0.4$.

4.3. Decision rules from disjunctive set-valued ordered decision tables

In a disjunctive set-valued ordered information system, the mapping $f : U \times A \to V$ is not single-valued but set-valued. Hence, we modify the definition of a formula over a according to the dominance relation $R_A^{\vee \geqslant}$ as follows

$$\|(a, \geqslant, v_1)\| = \{x \in U | \max f(x, a) \geqslant v_1\},$$
$$\|(a, \leqslant, v_1)\| = \{x \in U | \min f(x, a) \leqslant v_1\}.$$

For convenience, we continue to use $f(x, a) \geqslant v_1$ (meaning that $\max f(x, a) \geqslant v_1$) and $f(x, a) \leqslant v_1$ (meaning that $\min f(x, a) \leqslant v_1$) to express the relationships among set values in disjunctive set-valued OIS. Like decision rules in [40], there are four types of decision rules to be considered:

(1) certain $\geqslant$-decision rules with the following syntax:
if $(f(x, a_1) \geqslant v_{a_1}) \wedge (f(x, a_2) \geqslant v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \geqslant v_{a_k}) \wedge (f(x, a_{k+1}) \leqslant v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \leqslant v_{a_p})$, then $x \in D_i^{\geqslant}$;
(2) possible $\geqslant$-decision rules with the following syntax:
if $(f(x, a_1) \geqslant v_{a_1}) \wedge (f(x, a_2) \geqslant v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \geqslant v_{a_k}) \wedge (f(x, a_{k+1}) \leqslant v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \leqslant v_{a_p})$, then x could belong to $D_i^{\geqslant}$;
(3) certain $\leqslant$-decision rules with the following syntax:
if $(f(x, a_1) \leqslant v_{a_1}) \wedge (f(x, a_2) \leqslant v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \leqslant v_{a_k}) \wedge (f(x, a_{k+1}) \geqslant v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \geqslant v_{a_p})$, then $x \in D_i^{\leqslant}$;
(4) possible $\leqslant$-decision rules with the following syntax:
if $(f(x, a_1) \leqslant v_{a_1}) \wedge (f(x, a_2) \leqslant v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \leqslant v_{a_k}) \wedge (f(x, a_{k+1}) \geqslant v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \geqslant v_{a_p})$, then x could belong to $D_i^{\leqslant}$;

where $O_1 = \{a_1, a_2, \ldots, a_k\} \subseteq C$, $O_2 = \{a_{k+1}, a_{k+2}, \ldots, a_p\} \subseteq C$, $C = O_1 \cup O_2$, O_1 with increasing preference and O_2 with decreasing preference, $(v_{a_1}, v_{a_2}, \ldots, v_{a_p}) \in V_{a_1} \times V_{a_2} \times \cdots \times V_{a_p}$, $i \leqslant r$.

Therefore, in a disjunctive set-valued ODT, for a given upward or downward union $D_i^{\geqslant}$ or $D_j^{\leqslant}$, $i, j \leqslant r$, the rules induced under a hypothesis that objects belonging to $\underline{R_A^{\vee \geqslant}}(D_i^{\geqslant})$ or to $\underline{R_A^{\vee \leqslant}}(D_i^{\leqslant})$ are positive and all the others negative suggest the assignment of an object to "at least class D_i" or to "at most class D_j", respectively. Similarly, the rules induced under a hypothesis that objects belonging to $\overline{R_A^{\vee \geqslant}}(D_i^{\geqslant})$ or to $\overline{R_A^{\vee \leqslant}}(D_i^{\leqslant})$ are positive and all the others negative suggest the assignment of an object could belongs to "at least class D_i" or to "at most class D_j", respectively.

Now we employ an example to illustrate disjunctive set-valued ODT and decision rules extracted from this type of ODT in the following.

Example 4.3. Let us consider a disjunctive set-valued ODT, constructed from the disjunctive set-valued OIS in Table 3 and extended by decision attributes d as shown in Table 6.

From Table 6, it is easy to see that $\mathbf{D} = \{D_1, D_2\}$, where

$$D_1 = \{x_1, x_5, x_7, x_9, x_{10}\}, \quad D_2 = \{x_2, x_3, x_4, x_6, x_8\}.$$

In this disjunctive ordered decision table, because only two decision classes are considered, we have $D_1^{\geqslant} = D_1$ and $D_2^{\leqslant} = D_2$.

Table 6
A "$\bigvee$" set-valued ordered decision table.

U	a_1	a_2	a_3	a_4	a_5	d
x_1	{1}	{0,1}	{0}	{1,2}	{2}	2
x_2	{0,1}	{2}	{1,2}	{0}	{0}	1
x_3	{0}	{1,2}	{1}	{0,1}	{0}	1
x_4	{0}	{1}	{1}	{1}	{0,2}	1
x_5	{2}	{1}	{0,1}	{0}	{1}	2
x_6	{0,2}	{1}	{0,1}	{0}	{1}	1
x_7	{1}	{0,2}	{0,1}	{1}	{2}	2
x_8	{0}	{2}	{1}	{0}	{0,1}	1
x_9	{1}	{0,1}	{0,2}	{1}	{2}	2
x_{10}	{1}	{1}	{2}	{0,1}	{2}	2

From Definition 4.1, we obtain

$$\underline{R_C^{\vee\geqslant}}(D_1^{\geqslant}) = \{x_1, x_7, x_9, x_{10}\},$$
$$\overline{R_C^{\vee\geqslant}}(D_1^{\geqslant}) = \{x_1, x_5, x_6, x_7, x_9, x_{10}\},$$
$$Bn_C(D_1^{\geqslant}) = \{x_5, x_6\}.$$

Now we compute the dominated class of each object with respect to C in this decision table.

$$[x_1]_C^{\vee\leqslant} = \{x_1, x_6, x_7, x_9\}, [x_2]_C^{\vee\leqslant} = \{x_2, x_3, x_8\}, [x_3]_C^{\vee\leqslant} = \{x_2, x_3, x_4, x_8\},$$
$$[x_4]_C^{\vee\leqslant} = \{x_3, x_4, x_6\}, [x_5]_C^{\vee\leqslant} = [x_6]_C^{\vee\leqslant} = \{x_3, x_5, x_6\},$$
$$[x_7]_C^{\vee\leqslant} = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_9\}, [x_8]_C^{\vee\leqslant} = \{x_2, x_3, x_6, x_8\},$$
$$[x_9]_C^{\vee\leqslant} = [x_{10}]_C^{\vee\leqslant} = \{x_1, x_3, x_4, x_6, x_7, x_9, x_{10}\}.$$

From Definition 4.2 we easily infer that

$$\underline{R_C^{\vee\geqslant}}(D_2^{\leqslant}) = \{x_2, x_3, x_4, x_8\},$$
$$\overline{R_C^{\vee\geqslant}}(D_2^{\leqslant}) = \{x_2, x_3, x_4, x_5, x_6, x_8\},$$
$$Bn_C(D_2^{\leqslant}) = \{x_5, x_6\}.$$

One can obtain the following set of decision rules from the considered disjunctive ordered decision table:

$r_1 : (a_1, \geqslant, 1) \wedge (a_2, \geqslant, 0) \wedge (a_3, \geqslant, 0) \wedge (a_4, \geqslant, 0) \wedge (a_5, \geqslant, 2) \to (d, \geqslant, 2)$//supported by objects x_1, x_7, x_9;
$r_2 : (a_1, \geqslant, 1) \wedge (a_2, \geqslant, 1) \wedge (a_3, \geqslant, 2) \wedge (a_4, \geqslant, 0) \wedge (a_5, \geqslant, 2) \to (d, \geqslant, 2)$//supported by objects x_{10};
$r_3 : (a_1, \leqslant, 1) \wedge (a_2, \leqslant, 2) \wedge (a_3, \leqslant, 2) \wedge (a_4, \leqslant, 0) \wedge (a_5, \leqslant, 0) \to (d, \leqslant, 1)$//supported by objects x_2;
$r_4 : (a_1, \leqslant, 0) \wedge (a_2, \leqslant, 2) \wedge (a_3, \leqslant, 1) \wedge (a_4, \leqslant, 1) \wedge (a_5, \leqslant, 0) \to (d, \leqslant, 1)$//supported by objects x_3;
$r_5 : (a_1, \leqslant, 0) \wedge (a_2, \leqslant, 1) \wedge (a_3, \leqslant, 1) \wedge (a_4, \leqslant, 1) \wedge (a_5, \leqslant, 1) \to (d, \leqslant, 1)$//supported by objects x_4;
$r_6 : (a_1, \leqslant, 0) \wedge (a_2, \leqslant, 2) \wedge (a_3, \leqslant, 1) \wedge (a_4, \leqslant, 0) \wedge (a_5, \leqslant, 1) \to (d, \leqslant, 1)$//supported by objects x_8;
$r_7 : (a_1, =, 2) \wedge (a_2, =, 1) \wedge (a_3, =, \{0, 1\}) \wedge (a_4, =, 0) \wedge (a_5, =, 1) \to (d, \leqslant, 1) \vee (d, \geqslant, 2)$//supported by objects x_5;
$r_8 : (a_1, =, \{0, 2\}) \wedge (a_2, =, 1) \wedge (a_3, =, \{0, 1\}) \wedge (a_4, =, 0) \wedge (a_5, =, 1) \to (d, \leqslant, 1) \vee (d, \geqslant, 2)$//supported by objects x_6.

where r_1, r_2 are certain $\geqslant$-decision rules, r_3, r_4, r_5, r_6 are certain $\leqslant$-decision rules, while r_7, r_8 are not only possible $\geqslant$-decision rules but also possible $\leqslant$-decision rules. One can obtain three factors of the decision rule r_1

$$r_1 : (a_1, \geqslant, 1) \wedge (a_2, \geqslant, 0) \wedge (a_3, \geqslant, 0) \wedge (a_4, \geqslant, 0) \wedge (a_5, \geqslant, 2) \to (d, \geqslant, 2).$$

Here $card(\|\phi\|) = 3, card(\|\phi \wedge \varphi\|) = 3$, $card(\|\varphi\|) = 5$ and $|U| = 10$; hence $cer(r_1) = 1$, $sup(r_1) = 0.3$ and $cov(r_1) = 0.6$.

For any decision rule from a set-valued ODT, it should be minimal. Because a decision rule is an implication, by a minimal decision rule we understand such an implication to be that there is no other implication with an antecedent of at most the same weakness (in other words, a rule using a subset of elementary condition or/and weaker elementary conditions) and a consequent of at least the same strength (in other words, a rule assigning objects to the same union or sub-union of class). Hence, it is necessary to reduce some dispensable criterions in the condition part of a given set-valued ODT. In next section, we introduce criterion reduction approaches to a set-valued OIS and a set-valued ODT, respectively.

5. Criteria reductions in set-valued ordered information systems and set-valued ordered decision tables

To extract concise decision rules, it is necessary to reduce some criteria in the condition part of a given set-valued ODT. In this section, the approaches to the criterion reductions in a "$\bigwedge$"/"$\bigvee$" (conjunctive or disjunctive) set-valued ordered information system and a conjunctive/disjunctive set-valued ordered decision table are established, and several illustrative examples are employed to show their mechanisms as well.

5.1. Criteria reduction to conjunctive set-valued OIS and ODT

In the process of decision-making, sometimes there are some criteria that are redundant for ranking all objects according to a given dominance relation in ordered information systems. In other words, these criteria can be reduced from original criterion set on the basis of keeping the ordering of objects in ordered information systems. In this paper, we will introduce criterion reduction to describe the smallest criterion (attribute) subset that preserves the ordering of all objects in terms of a given dominance relation in set-valued ordered information systems and set-valued ordered decision tables.

Firstly, we investigate criteria reduction approach to a conjunctive set-valued ordered information system.

Definition 5.1. Let $S = (U, AT, V, f)$ be a conjunctive set-valued OIS and $A \subseteq AT$. If $R_A^{\wedge\geqslant} = R_{AT}^{\wedge\geqslant}$ and $R_B^{\wedge\geqslant} \neq R_{AT}^{\wedge\geqslant}$ for any $B \subset A$, then we call A is a criterion reduction of S.

It is obvious that a criterion reduction of a conjunctive set-valued OIS is a minimal attribute subset satisfying $R_A^{\wedge\geqslant} = R_{AT}^{\wedge\geqslant}$. An attribute $a \in AT$ is called dispensable with respect to $R_{AT}^{\wedge\geqslant}$ if $R_{AT}^{\wedge\geqslant} = R_{(AT-\{a\})}^{\wedge\geqslant}$; otherwise a is called indispensable. The set of all indispensable attributes ia called the core with respect to the dominance relation $R_{AT}^{\wedge\geqslant}$ and is denoted by $core(AT)$. An attribute in the core must be in every criterion reduction (like the case in complete/incomplete OIS, an OIS may have many reductions, denoted by $red(AT)$). Thus $core(AT) = \bigcap red(AT)$. The core may be an empty set.

From the above definition, it is easy to see that the criterion reduction is different from the classical attribute reduction from information systems in rough set theory. The attribute reduction is the smallest attribute subset that preserves the partition induced by original attributes in information systems, but the criterion reduction can preserve the ordering of all objects in terms of a given dominance relation in ordered information systems. Note that there must exist at least one criterion reduction in any set-valued ordered information systems.

Let $S = (U, AT, V, f)$ be a conjunctive set-valued OIS, $A \subseteq AT$. For convenient representation, let us use the notation

$$Dis_\wedge(x, y) = \left\{a \in A | (x, y) \notin R_a^{\wedge\geqslant}\right\}.$$

Then we call $Dis_\wedge(x, y)$ the discernibility attribute set between x and y, and

$$\mathbf{Dis}_\wedge = (Dis_\wedge(x, y) : x, y \in U)$$

the discernibility matrix of conjunctive set-valued OIS.

Clearly, for $\forall x, y \in U$ ***we have*** $Dis_\wedge(x, x) = \emptyset$ and $Dis_\wedge(x, y) \cap Dis_\wedge(y, x) = \emptyset$. The discernibility matrix gives the description of all of the criterion subsets, in which any two objects can be distinguished by a corresponding subset of criteria. Through using the discernibility matrix, one will be helpful for constructing and designing the approach to the criteria reduction in a conjunctive set-valued ordered information systems. It is deserved to point out that one can compute all criterion reducts of a conjunctive set-valued OIS by the discernibility matrix.

The following property provides a way in which reduction of criteria can be completed in conjunctive set-valued OIS.

Property 5.1. *Let $S = (U, AT, V, f)$ be a conjunctive set-valued OIS, $A \subseteq AT$, and $Dis_\wedge(x, y)$ the discernibility attributes set of S with respect to $R_{AT}^{\wedge\geqslant}$, then $R_{AT}^{\wedge\geqslant} = R_A^{\wedge\geqslant}$ iff $A \cap Dis_\wedge(x, y) \neq \emptyset$ ($Dis_\wedge(x, y) \neq \emptyset$).*

Proof. "$\Rightarrow$" Let $R_{AT}^{\wedge\geqslant} = R_A^{\wedge\geqslant}$, from the definition of this dominance relation, we obtain for arbitrary $x \in U$, $[x]_{AT}^{\wedge\geqslant} = [x]_A^{\wedge\geqslant}$ holds. If some $y \notin [x]_{AT}^{\wedge\geqslant}$, then $y \notin [x]_A^{\wedge\geqslant}$. Therefore, there exists $a \in A$ such that $(x, y) \notin [x]_{\{a\}}^{\wedge\geqslant}$. So one has $a \in Dis_\wedge(x, y)$. Hence, when $Dis_\wedge(x, y) \neq \emptyset$ we have $A \cap Dis_\wedge(x, y) \neq \emptyset$.

"$\Leftarrow$" From the definition of the discernibility attribute set, we know that if $(x, y) \notin [x]_{AT}^{\wedge\geqslant}$ for any $(x, y) \in U \times U$, then $Dis_\wedge(x, y) \neq \emptyset$. And since $A \cap Dis_\wedge(x, y) \neq \emptyset$, there exists $a \in A$ such that $a \in Dis_\wedge(x, y)$, i.e., $(x, y) \notin [x]_{\{a\}}^{\wedge\geqslant}$. So $(x, y) \notin [x]_A^{\wedge\geqslant}$. Hence $R_{AT}^{\wedge\geqslant} \supseteq R_A^{\wedge\geqslant}$. On the other hand, from $A \subseteq AT$ it follows that $R_{AT}^{\wedge\geqslant} \subseteq R_A^{\wedge\geqslant}$. Hence, one has $R_{AT}^{\wedge\geqslant} = R_A^{\wedge\geqslant}$.

This completes the proof. □

Definition 5.2. Let $S = (U, AT, V, f)$ be a conjunctive set-valued OIS, $A \subseteq AT$ and $Dis_\wedge(x, y)$ the discernibility attributes set of S with respect to $R_{AT}^{\wedge\geqslant}$. Denoted by

$$M_\wedge = \bigwedge \left\{\bigvee \{a : a \in Dis_\wedge(x, y)\} : x, y \in U\right\},$$

then $M_\wedge$ is referred to as the discernibility function.

Through using the discernibility function, one can design the approach to the criterion reduction in a conjunctive set-valued OIS as follows.

Property 5.2. *Let $S = (U, AT, V, f)$ be a conjunctive set-valued OIS. The minimal disjunctive normal form of discernibility function M is*

$$M_\wedge = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_{i_s}\right).$$

Denoted by $B_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$, then $\{B_k : k = 1, 2, \ldots, t\}$ are the set of all criterion reductions of this system.

Proof. It follows directly from Property 5.1 and the definition of minimal disjunctive normal form of the discernibility function. □

Property 5.2 provides a practical approach to criterion reduction in a conjunctive set-valued OIS.

In the following, an illustrative example is employed to illustrate the mechanism of this approach.

Example 5.1. Continuation of Example 3.1. Compute all criteria reductions in Table 2.

Briefly, A, S, R and W will stand for Audition, Spoken language, Reading and Writing, respectively. One obtains the discernibility matrix of this system (see Table 7).

Hence, we have

$$M_{\wedge} = (A \vee S \vee W) \wedge (A \vee S) \wedge (A \vee S \vee R \vee W) \wedge R \wedge (A \vee R \vee W) \wedge S \wedge W \wedge (S \vee R \vee W) \wedge (A \vee W) \wedge (R \vee W) \wedge (A \vee S \vee R) \wedge (A \vee R) \wedge (S \vee W) \wedge (S \vee R) = S \wedge R \wedge W.$$

Therefore, $\{S, R, W\}$ is a unique criterion reduction for this system, that is, the attribute Audition can be eliminated from Table 2.

As follows, we research on criterion reduction of conjunctive set-valued ODT for mining more briefer decision rules.

Let $S = (U, C \cup \{d\}, V, f)$ be a conjunctive set-valued ODT and d is an overall preference of objects. Denoted by

$$R^{\geqslant}_{\{d\}} = \{(x, y) : f(x, d) \geqslant f(y, d),$$

where $R^{\geqslant}_{\{d\}}$ is a dominance relation of decision attribute d. If $R^{\wedge\geqslant}_{C} \subseteq R^{\geqslant}_{\{d\}}$, then S is called consistent; otherwise it is inconsistent. For example, Table 5 is a consistent conjunctive set-valued ODT in fact. In this paper, we only deal with criterion reduction of a consistent conjunctive set-valued ODT.

Definition 5.3. Let $S = (U, C \cup \{d\}, V, f)$ be a consistent conjunctive set-valued ODT, $A \subseteq C$. If $R^{\wedge\geqslant}_{A} \subseteq R^{\geqslant}_{\{d\}}$ and $R^{\wedge\geqslant}_{B} \not\subseteq R^{\geqslant}_{\{d\}}$ for any $B \subset A$, then we call A is a relative criterion reduction of S.

Similarly to the idea of reducts of incomplete ODT in [36], we denote by $D^{*} = \{(x, y) : f(x, d) < f(y, d)\}$, and denote by

$$Dis^{*}_{\wedge}(x, y) = \begin{cases} \{a \in C : (x, y) \notin R^{\wedge\geqslant}_{\{a\}}\}, (x, y) \notin D^{*}; \\ \emptyset, (x, y) \in D^{*}. \end{cases}$$

Then $Dis^{*}_{\wedge}(x, y)$ is called a discernibility set for objects x and y, and $\mathbf{Dis}^{*}_{\wedge} = (Dis^{*}_{\wedge}(x, y) : x, y \in U)$ is called a discernibility matrix for the conjunctive set-valued ODT S.

Similar to the conjunctive set-valued OIS, one can show the following property.

Property 5.3. *Let $S = (U, C \cup \{d\}, V, f)$ be a conjunctive set-valued ODT, $A \subseteq C$ and $Dis^{*}_{\wedge}(x, y)$ the discernibility attributes set of S with respect to $R^{\geqslant}_{\{d\}}$, then $R^{\wedge\geqslant}_{A} \subseteq R^{\geqslant}_{\{d\}}$ iff $A \cap Dis^{*}_{\wedge}(x, y) \neq \emptyset$ ($Dis^{*}_{\wedge}(x, y) \neq \emptyset$).*

Proof. The proof is similar to the proof of Property 5.1. □

Definition 5.4. Let $S = (U, C \cup \{d\}, V, f)$ be a conjunctive set-valued ODT, $A \subseteq C$ and $Dis^{*}_{\wedge}(x, y)$ the discernibility attributes set of S with respect to $R^{\geqslant}_{\{d\}}$. Denoted by

$$M^{*}_{\wedge} = \bigwedge \left\{ \bigvee \{a : a \in Dis^{*}_{\wedge}(x, y)\} : x, y \in U \right\},$$

then $M^{*}_{\wedge}$ is referred to as the discernibility function.

By using the discernibility function, one can design the approach to the relative criterion reduction in a conjunctive set-valued ODT as follows.

Table 7
The discernibility matrix of Table 2.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	∅	A, S, W	A, S	A, S	A, S	A, S, R, W	A, S, R, W	A, S, R, W	A, S	A, S, W
x_2	∅	∅	∅	∅	∅	R	R	R	∅	∅
x_3	∅	A, S, W	∅	A, S	A, S	A, R, W	A, S, R, W	A, S, R, W	A, S	A, S, W
x_4	W	A, S, W	A, S, W	∅	A, S	S, R, W	A, S, R, W	S, R, W	A, W	W
x_5	A, S, W	A, S, W	A, S, W	A, S	∅	R, W	A, S, R, W	A, R, W	W	A, S, W
x_6	A, S, R, W	A, S, R, W	A, S, R, W	A, S, R	A, R	∅	A, S, R, W	A, S, R, W	A, S, R, W	A, S, R
x_7	R	R	R	R	R	R	∅	R	R	R
x_8	S, W	A, S, W	A, S, W	S, W	A, W	W	A, S, W	∅	A, W	S, W
x_9	A, S	A, S, W	A, S	A, S	S	S, R, W	A, S, R, W	A, S, R, W	∅	A, S, W
x_{10}	W	A, S, W	A, S, W	∅	A, S	S, R	A, S, R, W	S, R, W	A, W	∅

Property 5.4. *Let $S = (U, C \cup \{d\}, V, f)$ be a conjunctive set-valued ODT. The minimal disjunctive normal form of discernibility function $M^*_\wedge$ is*

$$M^*_\wedge = \bigvee_{k=1}^{t}\left(\bigwedge_{s=1}^{q_k} a_{i_s}\right).$$

Denoted by $B_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$, then $\{B_k : k = 1, 2, \ldots, t\}$ are the set of all relative criterion reductions of this system.

Proof. It follows directly from Property 5.3 and the definition of minimal disjunctive normal form of the discernibility function. □

Property 5.4 provides a practical approach to relative criterion reduction in a conjunctive set-valued ODT.

Example 5.2. Compute all relative criterion reductions in Table 5.

Table 8 is a discernibility matrix of this consistent decision table, where values of $Dis^*_\wedge(x_i, x_j)$ for any pair (x_i, x_j) of objects from U are placed.

From Table 8, we have

$$M^*_\wedge = (A \vee S) \wedge (A \vee S \vee R \vee W) \wedge R \wedge (A \vee S \vee W) \wedge (A \vee R \vee W) \wedge S \wedge W \wedge (S \vee R \vee W) \wedge (A \vee W) \wedge (R \vee W) \wedge (S \vee W) \wedge (A \vee S \vee R) \wedge (S \vee R) = S \wedge R \wedge W.$$

Hence, there is a unique relative criterion reduction $\{S, R, W\}$ in the consistent conjunctive set-valued ordered decision table. From this example, we know that the condition attributes Spoken language, Reading and Writing are all indispensable for this decision table. Through the relative criterion reduction, one can obtain more briefer decision rules as follows:

r_1 : (Spoken language, $\supseteq, \{E, F\}) \wedge$ (Reading, $\supseteq, \{F, G\}) \wedge$ (Writing, $\supseteq, \{F, G\}) \rightarrow (d, \geqslant, Good)$//supported by objects x_2, x_3;
r_2 : (Spoken language, $\supseteq, \{E, F, G\}) \wedge$ (Reading, $\supseteq, \{E, G\}) \wedge$ (Writing, $\supseteq, \{E, F, G\}) \rightarrow (d, \geqslant, Good)$//supported by objects x_7;
r_3 : (Spoken language, $\supseteq, \{F, G\}) \wedge$ (Reading, $\supseteq, \{E, F, G\}) \wedge$ (Writing, $\supseteq, \{E, G\}) \rightarrow (d, \geqslant, Good)$//supported by objects x_8;
r_4 : (Spoken language, $\supseteq, \{E, G\}) \wedge$ (Reading, $\supseteq, \{F, G\}) \wedge$ (Writing, $\supseteq, \{E, F\}) \rightarrow (d, \geqslant, Good)$//supported by objects x_2, x_{10};
r_5 : (Spoken language, $\subseteq, \{E\}) \wedge$ (Reading, $\subseteq, \{F, G\}) \wedge$ (Writing, $\subseteq, \{F, G\}) \rightarrow (d, \leqslant, Poor)$//supported by objects x_1;
r_6 : (Spoken language, $\subseteq, \{E, G\}) \wedge$ (Reading, $\subseteq, \{F, G\}) \wedge$ (Writing, $\subseteq, \{F\}) \rightarrow (d, \leqslant, Poor)$//supported by objects x_4;
r_7 : (Spoken language, $\subseteq, \{F, G\}) \wedge$ (Reading, $\subseteq, \{F, G\}) \wedge$ (Writing, $\subseteq, \{F\}) \rightarrow (d, \leqslant, Poor)$//supported by objects x_5;
r_8 : (Spoken language, $\subseteq, \{F\}) \wedge$ (Reading, $\subseteq, \{E, F\}) \wedge$ (*Writing*, $\subseteq, \{E, F\}) \rightarrow (d, \leqslant, Poor)$//supported by objects x_6;
r_9 : (Spoken language, $\subseteq, \{G\}) \wedge$ (Reading, $\subseteq, \{F, G\}) \wedge$ (Writing, $\subseteq, \{F, G\}) \rightarrow (d, \leqslant, Poor)$//supported by objects x_9.

where r_1, r_2, r_3, r_4 are certain $\supseteq$-decision rules, r_5, r_6, r_7, r_8, r_9 are certain $\subseteq$-decision rules.

5.2. Criterion reduction to disjunctive set-valued OIS and ODT

First, we give the definition of a criterion reduction of a disjunctive set-valued ordered information system.

Definition 5.5. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS and $A \subseteq AT$. If $R^{\vee\geqslant}_A = R^{\vee\geqslant}_{AT}$ and $R^{\vee\geqslant}_B \neq R^{\vee\geqslant}_{AT}$ for any $B \subset A$, then we call A is a criterion reduction of S.

It is clear that a criterion reduction of a disjunctive set-valued OIS is a minimal attribute subset satisfying $R^{\vee\geqslant}_A = R^{\vee\geqslant}_{AT}$. An attribute $a \in AT$ is called dispensable with respect to $R^{\vee\geqslant}_{AT}$ if $R^{\vee\geqslant}_{AT} = R^{\vee\geqslant}_{(AT-\{a\})}$; otherwise a is called indispensable. The set of all indispensable attributes ia called the core with respect to the dominance relation $R^{\vee\geqslant}_{AT}$ and is denoted by $core(AT)$. An attribute in the core must be in every criterion reduction (like the case in complete/incomplete OIS, an OIS may have many reductions, denoted by $red(AT)$). Thus $core(AT) = \bigcap red(AT)$. The core may be an empty set.

Table 8
The discernibility matrix of Table 5.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	∅	∅	∅	A, S	A, S	A, S, R, W	∅	∅	A, S	∅
x_2	∅	∅	∅	∅	∅	R	R	R	∅	∅
x_3	∅	A, S, W	∅	A, S	A, S	A, R, W	A, S, R, W	A, S, R, W	A, S	A, S, W
x_4	W	∅	∅	∅	A, S	S, R, W	∅	∅	A, W	∅
x_5	A, S, W	∅	∅	A, S	∅	R, W	∅	∅	W	∅
x_6	A, S, R, W	∅	∅	A, S, R	A, R	∅	∅	∅	A,S,R,W	∅
x_7	R	R	R	R	R	R	∅	R	R	R
x_8	S, W	A, S, W	A, S, W	S, W	A, W	W	A, S, W	∅	A, W	S, W
x_9	A, S	∅	∅	A, S	S	S, R, W	∅	∅	∅	∅
x_{10}	W	A, S, W	A, S, W	∅	A, S	S, R	A, S, R, W	S, R, W	A, W	∅

Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS and $A \subseteq AT$. For convenient representation, denoted by

$$Dis_{\vee}(x,y) = \{a \in A | (x,y) \notin R_a^{\vee \geqslant}\},$$

then we call $Dis_{\vee}(x,y)$ the discernibility attribute set between x and y, and

$$\mathbf{Dis}_{\vee} = (Dis_{\vee}(x,y) : x,y \in U)$$

the discernibility matrix of disjunctive set-valued OIS. Clearly, for $\forall x,y \in U$ we have $Dis_{\vee}(x,y) \cap Dis_{\vee}(y,x) = \emptyset$.

The following property provides a judgement method of a criterion reduction of disjunctive set-valued OIS.

Property 5.5. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$ and $Dis_{\vee}(x,y)$ the discernibility attributes set of S with respect to $R_{AT}^{\vee \geqslant}$, then $R_{AT}^{\vee \geqslant} = R_A^{\vee \geqslant}$ iff $A \cap Dis_{\vee}(x,y) \neq \emptyset$ ($Dis_{\vee}(x,y) \neq \emptyset$).*

Proof. "⇒" Let $R_{AT}^{\vee \geqslant} = R_A^{\vee \geqslant}$, from the definition of the dominance relation, we have for arbitrary $x \in U$, $[x]_{AT}^{\vee \geqslant} = [x]_A^{\vee \geqslant}$ holds. If some $y \notin [x]_{AT}^{\vee \geqslant}$, then $y \notin [x]_A^{\vee \geqslant}$. Therefore, there exists $a \in A$ such that $(x,y) \notin [x]_{\{a\}}^{\vee \geqslant}$. So one has $a \in Dis_{\vee}(x,y)$. Hence, when $Dis_{\vee}(x,y) \neq \emptyset$ we have $A \cap Dis_{\vee}(x,y) \neq \emptyset$.

"⇐" From the definition of the discernibility attribute set, we know that if $(x,y) \notin [x]_{AT}^{\vee \geqslant}$ for any $(x,y) \in U \times U$, then $Dis_{\vee}(x,y) \neq \emptyset$. And since $A \cap Dis_{\vee}(x,y) \neq \emptyset$, there exists $a \in A$ such that $a \in Dis_{\vee}(x,y)$, i.e., $(x,y) \notin [x]_{\{a\}}^{\vee \geqslant}$. So $(x,y) \notin [x]_A^{\vee \geqslant}$. Hence $R_{AT}^{\vee \geqslant} \supseteq R_A^{\vee \geqslant}$. On the other hand, it follows from $A \subseteq AT$ that $R_{AT}^{\vee \geqslant} \subseteq R_A^{\vee \geqslant}$. Hence, one has $R_{AT}^{\vee \geqslant} = R_A^{\vee \geqslant}$.

This completes the proof. □

Definition 5.6. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$ and $Dis_{\vee}(x,y)$ the discernibility attributes set of S with respect to $R_{AT}^{\vee \geqslant}$. Denoted by

$$M_{\vee} = \bigwedge \left\{ \bigvee \{a : a \in Dis_{\vee}(x,y)\} : x,y \in U \right\},$$

then M is referred to as the discernibility function.

By using the discernibility function, one can give the approach to the criterion reduction in a disjunctive set-valued OIS as follows.

Property 5.6. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS. The minimal disjunctive normal form of discernibility function $M_{\vee}$ is*

$$M_{\vee} = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_{i_s} \right).$$

Denoted by $B_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$, then $\{B_k : k = 1, 2, \ldots, t\}$ are the set of all criterion reductions of this system.

Proof. It follows directly from Property 5.5 and the definition of minimal disjunctive normal form of the discernibility function. □

Property 5.6 provides a practical approach to criterion reduction in a disjunctive set-valued OIS.

In the following, an illustrative example is employed to analyze the mechanism of this approach.

Example 5.3. Here we continue Example 3.3. Compute all criterion reductions in Table 3.

By computing, one can obtain the discernibility matrix of this system (see Table 9).

Hence, we have

$$\begin{aligned} M_{\vee} &= (a_2 \vee a_3) \wedge a_3 \wedge a_1 \wedge (a_4 \vee a_5) \wedge a_4 \wedge (a_1 \vee a_5) \wedge a_5 \wedge (a_1 \vee a_3 \vee a_5) \wedge (a_1 \vee a_3) \wedge (a_3 \vee a_5) \wedge (a_1 \vee a_4 \wedge a_5) \wedge a_2 \\ &= a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5. \end{aligned}$$

Table 9
The discernibility matrix of Table 3.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$\emptyset$	$\{a_2,a_3\}$	$\{a_3\}$	$\{a_3\}$	$\{a_1\}$	$\emptyset$	$\emptyset$	$\{a_2,a_3\}$	$\emptyset$	$\{a_3\}$
x_2	$\{a_4,a_5\}$	$\emptyset$	$\{\emptyset\}$	$\{a_4\}$	$\{a_1,a_5\}$	$\{a_5\}$	$\{a_4,a_5\}$	$\emptyset$	$\{a_4,a_5\}$	$\{a_5\}$
x_3	$\{a_1,a_5\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_1,a_5\}$	$\{a_5\}$	$\{a_1,a_5\}$	$\emptyset$	$\{a_1,a_5\}$	$\{a_1,a_3,a_5\}$
x_4	$\{a_1\}$	$\{a_2\}$	$\emptyset$	$\emptyset$	$\{a_1\}$	$\emptyset$	$\{a_1\}$	$\{a_2\}$	$\{a_1\}$	$\{a_1,a_3\}$
x_5	$\{a_4,a_5\}$	$\{a_2\}$	$\emptyset$	$\{a_4\}$	$\emptyset$	$\emptyset$	$\{a_4,a_5\}$	$\{a_2\}$	$\{a_4,a_5\}$	$\{a_3,a_5\}$
x_6	$\{a_4,a_5\}$	$\{a_2\}$	$\emptyset$	$\{a_4\}$	$\emptyset$	$\emptyset$	$\{a_4,a_5\}$	$\{a_2\}$	$\{a_4,a_5\}$	$\{a_3,a_5\}$
x_7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_1\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_3\}$
x_8	$\{a_1,a_4,a_5\}$	$\emptyset$	$\emptyset$	$\{a_4\}$	$\{a_1\}$	$\emptyset$	$\{a_1,a_4,a_5\}$	$\emptyset$	$\{a_1,a_4,a_5\}$	$\{a_1,a_3,a_5\}$
x_9	$\emptyset$	$\{a_2\}$	$\emptyset$	$\emptyset$	$\{a_1\}$	$\emptyset$	$\emptyset$	$\{a_2\}$	$\emptyset$	$\emptyset$
x_{10}	$\emptyset$	$\{a_2\}$	$\emptyset$	$\emptyset$	$\{a_1\}$	$\emptyset$	$\emptyset$	$\{a_2\}$	$\emptyset$	$\emptyset$

Therefore, $\{a_1, a_2, a_3, a_4, a_5\}$ is a unique criterion reduction for this system, that is, any attribute cannot be eliminated from Table 2.

As follows, we examine how to find a criterion reduction from a given disjunctive set-valued ordered decision table. Let $S = (U, C \cup \{d\}, V, f)$ be a disjunctive set-valued ODT, d is an overall preference of objects. Denoted by

$$R^{\geqslant}_{\{d\}} = \{(x, y) : f(x, d) \geqslant f(y, d),$$

$R^{\geqslant}_{\{d\}}$ is a dominance relation of decision attribute d. If $R^{\vee\geqslant}_{C} \subseteq R^{\geqslant}_{\{d\}}$, then S is called consistent; otherwise it is inconsistent. For example, Table 10 is a consistent disjunctive set-valued ODT. Next, we deal with criterion reduction of a consistent disjunctive set-valued ODT.

Definition 5.7. Let $S = (U, C \cup \{d\}, V, f)$ be a consistent disjunctive set-valued ODT and $A \subseteq C$. If $R^{\vee\geqslant}_{A} \subseteq R^{\geqslant}_{\{d\}}$ and $R^{\vee\geqslant}_{B} \not\subseteq R^{\geqslant}_{\{d\}}$ for any $B \subset A$, then we call A is a relative criterion reduction of S.

Similarly to the idea of reducts of incomplete ODT in [36], we denote by $D^* = \{(x, y) : f(x, d) < f(y, d)\}$, and denote it by

$$Dis^*_{\vee}(x, y) = \begin{cases} \{a \in C : (x, y) \notin R^{\vee\geqslant}_{\{a\}}\}, (x, y) \notin D^*; \\ \emptyset, (x, y) \in D^*. \end{cases}$$

Then $Dis^*(x, y)$ is called a discernibility set for objects x and y, and $\mathbf{Dis}^*_{\vee} = (Dis^*_{\vee}(x, y) : x, y \in U)$ is called a discernibility matrix for the disjunctive set-valued ODT S.

Similarly to the disjunctive set-valued OIS, we can show the following property.

Property 5.7. *Let $S = (U, C \cup \{d\}, V, f)$ be a disjunctive set-valued ODT, $A \subseteq C$ and $Dis^*_{\vee}(x, y)$ the discernibility attributes set of S with respect to $R^{\geqslant}_{\{d\}}$, then $R^{\vee\geqslant}_{A} \subseteq R^{\geqslant}_{\{d\}}$ iff $A \cap Dis^*_{\vee}(x, y) \neq \emptyset$ ($Dis^*_{\vee}(x, y) \neq \emptyset$).*

Proof. The proof is similar to the proof of Property 5.5. □

Definition 5.8. Let $S = (U, C \cup \{d\}, V, f)$ be a disjunctive set-valued ODT, $A \subseteq C$ and $Dis^*(x, y)$ the discernibility attributes set of S with respect to $R^{\geqslant}_{\{d\}}$. Denoted by

$$M^*_{\vee} = \bigwedge \left\{ \bigvee \{a : a \in Dis^*_{\vee}(x, y)\} : x, y \in U \right\},$$

then $M^*_{\vee}$ is referred to as the discernibility function.

By using the discernibility function, one can design the relative criteria reduction in a disjunctive set-valued ODT.

Property 5.8. *Let $S = (U, C \cup \{d\}, V, f)$ be a disjunctive set-valued ODT. The minimal disjunctive normal form of discernibility function $M^*_{\vee}$ is*

$$M^*_{\vee} = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_{i_s} \right).$$

Denote by $B_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$. Then $\{B_k : k = 1, 2, \ldots, t\}$ is the set of all relative criteria reductions of this system.

Proof. It follows directly from Property 5.7 and the definition of minimal disjunctive normal form of the discernibility function. □

Property 5.8 leads to the method of relative criteria reductions in a disjunctive set-valued ODT.

Example 5.4. Compute all relative criteria reductions in Table 10.

Table 11 is a discernibility matrix of this consistent decision table, where values of $Dis^*(x_i, x_j)$ for any pair (x_i, x_j) of objects from U are placed. From Table 11, we have

$$M^*_{\vee} = a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3).$$

Table 10
A consistent "$\bigvee$" set-valued ordered decision table.

U	a_1	a_2	a_3	a_4	d
x_1	{0, 1}	{0}	{0}	{0, 1}	1
x_2	{0}	{3}	{2}	{0}	1
x_3	{1}	{3}	{2}	{0, 1}	1
x_4	{2}	{1, 2}	{1}	{1}	2
x_5	{2, 3}	{1}	{1}	{0, 1}	2
x_6	{3}	{0}	{0}	{1}	2

Table 11
The discernibility matrix of Table 10.

U	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$\emptyset$	$\{a_2, a_3\}$	$\{a_2, a_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
x_2	$\emptyset$	$\emptyset$	a_1	$\emptyset$	$\emptyset$	$\emptyset$
x_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	$\emptyset$	$\{a_2, a_3\}$	$\{a_2, a_3\}$	$\emptyset$	$\emptyset$	$\{a_1\}$
x_5	$\emptyset$	$\{a_2, a_3\}$	$\{a_2, a_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
x_6	$\emptyset$	$\{a_2, a_3\}$	$\{a_2, a_3\}$	$\{a_2, a_3\}$	$\{a_2, a_3\}$	$\emptyset$

Table 12
A practical conjunctive set-valued information system from the test for foreign language ability in Shanxi University.

Students	Audition	Spoken language	Reading	Writing
x_1	$\{E\}$	$\{E\}$	$\{F, G\}$	$\{F, G\}$
x_2	$\{E, F, G\}$	$\{E, F, G\}$	$\{F, G\}$	$\{E, F, G\}$
x_3	$\{F, G\}$	$\{F\}$	$\{F, G\}$	$\{F, G\}$
x_4	$\{E, F\}$	$\{E, G\}$	$\{F, G\}$	$\{F\}$
x_5	$\{F, G\}$	$\{F, G\}$	$\{F, G\}$	$\{F\}$
x_6	$\{F\}$	$\{F\}$	$\{E, F\}$	$\{E, F\}$
x_7	$\{E, F, G\}$	$\{E, F, G\}$	$\{E, G\}$	$\{E, F, G\}$
x_8	$\{F, G\}$	$\{F\}$	$\{F, G\}$	$\{F, G\}$
x_9	$\{E, G\}$	$\{G\}$	$\{F, G\}$	$\{F, G\}$
x_{10}	$\{E, F\}$	$\{E, G\}$	$\{F, G\}$	$\{E, F\}$
x_{11}	$\{F\}$	$\{E, F\}$	$\{F\}$	$\{G\}$
x_{12}	$\{E, F, G\}$	$\{E, G\}$	$\{E, F, G\}$	$\{E, G\}$
x_{13}	$\{E, F, G\}$	$\{G\}$	$\{E, G\}$	$\{F\}$
x_{14}	$\{F, G\}$	$\{F, G\}$	$\{F, G\}$	$\{F\}$
x_{15}	$\{E, F, G\}$	$\{E, F, G\}$	$\{F, G\}$	$\{E, F\}$
x_{16}	$\{E, G\}$	$\{F, G\}$	$\{F, G\}$	$\{E, G\}$
x_{17}	$\{E, G\}$	$\{F\}$	$\{E, F, G\}$	$\{E, G\}$
x_{18}	$\{E, F, G\}$	$\{F, G\}$	$\{F\}$	$\{E, F, G\}$
x_{19}	$\{F\}$	$\{E, F\}$	$\{F, G\}$	$\{E, F\}$
x_{20}	$\{E\}$	$\{E\}$	$\{F, G\}$	$\{F, G\}$
x_{21}	$\{F, G\}$	$\{F\}$	$\{F, G\}$	$\{F, G\}$
x_{22}	$\{F\}$	$\{E, F\}$	$\{F\}$	$\{G\}$
x_{23}	$\{E, F\}$	$\{E, G\}$	$\{F, G\}$	$\{F\}$
x_{24}	$\{F\}$	$\{E, F\}$	$\{F\}$	$\{G\}$
x_{25}	$\{E, F, G\}$	$\{E, G\}$	$\{E, F, G\}$	$\{E, G\}$
x_{26}	$\{F, G\}$	$\{F, G\}$	$\{F, G\}$	$\{F\}$
x_{27}	$\{F, G\}$	$\{F\}$	$\{F, G\}$	$\{F, G\}$
x_{28}	$\{E, F\}$	$\{E, G\}$	$\{F, G\}$	$\{F\}$
x_{29}	$\{E, F, G\}$	$\{F, G\}$	$\{F\}$	$\{E, F, G\}$
x_{30}	$\{E, G\}$	$\{F\}$	$\{F\}$	$\{E, F, G\}$
x_{31}	$\{F, G\}$	$\{F, G\}$	$\{E, G\}$	$\{F, G\}$
x_{32}	$\{E, F, G\}$	$\{E, G\}$	$\{E, G\}$	$\{F, G\}$
x_{33}	$\{E, F, G\}$	$\{F, G\}$	$\{F, G\}$	$\{F, G\}$
x_{34}	$\{E, F, G\}$	$\{F, G\}$	$\{E, F, G\}$	$\{E, G\}$
x_{35}	$\{F\}$	$\{E, F\}$	$\{F\}$	$\{G\}$
x_{36}	$\{E, F, G\}$	$\{F, G\}$	$\{F\}$	$\{E, F, G\}$
x_{37}	$\{F, G\}$	$\{F\}$	$\{F, G\}$	$\{F, G\}$
x_{38}	$\{F, G\}$	$\{F, G\}$	$\{F, G\}$	$\{F\}$
x_{39}	$\{E, F\}$	$\{F\}$	$\{E, F, G\}$	$\{F\}$
x_{40}	$\{F\}$	$\{E, F\}$	$\{F\}$	$\{G\}$
x_{41}	$\{E, F, G\}$	$\{F, G\}$	$\{F\}$	$\{E, F, G\}$
x_{42}	$\{E, F, G\}$	$\{E, F, G\}$	$\{E, F, G\}$	$\{E, G\}$
x_{43}	$\{E, F\}$	$\{F\}$	$\{E, F, G\}$	$\{F\}$
x_{44}	$\{F\}$	$\{F\}$	$\{F, G\}$	$\{E, F\}$
x_{45}	$\{E, F, G\}$	$\{E, F, G\}$	$\{F, G\}$	$\{E, F, G\}$
x_{46}	$\{E, G\}$	$\{E, G\}$	$\{GF\}$	$\{F, G\}$
x_{47}	$\{E, F, G\}$	$\{E, F, G\}$	$\{E, F, G\}$	$\{E, G\}$
x_{48}	$\{E, F\}$	$\{E, G\}$	$\{F, G\}$	$\{F\}$
x_{49}	$\{E, F, G\}$	$\{F, G\}$	$\{E, F, G\}$	$\{E, G\}$
x_{50}	$\{F, G\}$	$\{F\}$	$\{F, G\}$	$\{F, G\}$

Hence, there are two relative criterion reductions $\{a_1, a_2\}$ and $\{a_1, a_3\}$ in the consistent disjunctive set-valued ordered decision table. From this example, we know that a_1 is indispensable for this decision table because of $\{a_1, a_2\} \cap \{a_1, a_3\} = \{a_1\}$, and the attribute a_4 can be eliminated from the decision table. Obviously, from the relative criterion reductions, we can extract concise decision rules from this type of decision tables.

Table 13
The discernibility matrix of Table 12.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	...	...	...	x_{48}	x_{49}	x_{50}
x_1	∅	A, S, W	A,S	A, S	A, S	...	...	...	A, S	A, S, W, R	A, S
x_2	∅	∅	∅	∅	∅	...	...	...	∅	R	∅
x_3	A, S	A, S, W	∅	A, S	S	...	...	...	S	A, S, R, W	∅
x_4	W	A, S, R	A, S, W	∅	A, S	...	...	...	S, W	A, R, S, W	A, S, W
x_5	A, S, W	A, S, W	W	A, S	∅	...	...	...	S, W	A, R, W	W
⋮						⋱					
⋮							⋱				
⋮								⋱			
x_{48}	W	A, S, R	A, S, W	∅	A, S	...	...	...	S, W	A, S, R, W	A, S, W
x_{49}	S, W	S, W	W	S, W	W	...	...	...	S	∅	W
x_{50}	A, S	A, S, W	∅	A, S	S	...	...	...	S	A, S, R, W	∅

Table 14
A practical conjunctive set-valued decision table from the test for foreign language ability in Shanxi University.

Students	Audition	Spoken language	Reading	Writing	Evaluation
x_1	$\{E\}$	$\{E\}$	$\{F,G\}$	$\{F,G\}$	*Poor*
x_2	$\{E,F,G\}$	$\{E,F,G\}$	$\{F,G\}$	$\{E,F,G\}$	*Good*
x_3	$\{F,G\}$	$\{F\}$	$\{F,G\}$	$\{F,G\}$	*Good*
x_4	$\{E,F\}$	$\{E,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_5	$\{F,G\}$	$\{F,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_6	$\{F\}$	$\{F\}$	$\{E,F\}$	$\{E,F\}$	*Poor*
x_7	$\{E,F,G\}$	$\{E,F,G\}$	$\{E,G\}$	$\{E,F,G\}$	*Good*
x_8	$\{F,G\}$	$\{F\}$	$\{F,G\}$	$\{F,G\}$	*Good*
x_9	$\{E,G\}$	$\{G\}$	$\{F,G\}$	$\{F,G\}$	*Poor*
x_{10}	$\{E,F\}$	$\{E,G\}$	$\{F,G\}$	$\{E,F\}$	*Good*
x_{11}	$\{F\}$	$\{E,F\}$	$\{F\}$	$\{G\}$	*Good*
x_{12}	$\{E,F,G\}$	$\{E,G\}$	$\{E,F,G\}$	$\{E,G\}$	*Poor*
x_{13}	$\{E,F,G\}$	$\{G\}$	$\{E,G\}$	$\{F\}$	*Poor*
x_{14}	$\{F,G\}$	$\{F,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_{15}	$\{E,F,G\}$	$\{E,F,G\}$	$\{F,G\}$	$\{E,F\}$	*Good*
x_{16}	$\{E,G\}$	$\{F,G\}$	$\{F,G\}$	$\{E,G\}$	*Good*
x_{17}	$\{E,G\}$	$\{F\}$	$\{E,F,G\}$	$\{E,G\}$	*Poor*
x_{18}	$\{E,F,G\}$	$\{F,G\}$	$\{F\}$	$\{E,F,G\}$	*Poor*
x_{19}	$\{F\}$	$\{E,F\}$	$\{F,G\}$	$\{E,F\}$	*Poor*
x_{20}	$\{E\}$	$\{E\}$	$\{F,G\}$	$\{F,G\}$	*Good*
x_{21}	$\{F,G\}$	$\{F\}$	$\{F,G\}$	$\{F,G\}$	*Good*
x_{22}	$\{F\}$	$\{E,F\}$	$\{F\}$	$\{G\}$	*Good*
x_{23}	$\{E,F\}$	$\{E,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_{24}	$\{F\}$	$\{E,F\}$	$\{F\}$	$\{G\}$	*Good*
x_{25}	$\{E,F,G\}$	$\{E,G\}$	$\{E,F,G\}$	$\{E,G\}$	*Poor*
x_{26}	$\{F,G\}$	$\{F,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_{27}	$\{F,G\}$	$\{F\}$	$\{F,G\}$	$\{F,G\}$	*Good*
x_{28}	$\{E,F\}$	$\{E,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_{29}	$\{E,F,G\}$	$\{F,G\}$	$\{F\}$	$\{E,F,G\}$	*Poor*
x_{30}	$\{E,G\}$	$\{F\}$	$\{F\}$	$\{E,F,G\}$	*Good*
x_{31}	$\{F,G\}$	$\{F,G\}$	$\{E,G\}$	$\{F,G\}$	*Good*
x_{32}	$\{E,F,G\}$	$\{E,G\}$	$\{E,G\}$	$\{F,G\}$	*Good*
x_{33}	$\{E,F,G\}$	$\{F,G\}$	$\{F,G\}$	$\{F,G\}$	*Poor*
x_{34}	$\{E,F,G\}$	$\{F,G\}$	$\{E,F,G\}$	$\{E,G\}$	*Poor*
x_{35}	$\{F\}$	$\{E,F\}$	$\{F\}$	$\{G\}$	*Good*
x_{36}	$\{E,F,G\}$	$\{F,G\}$	$\{F\}$	$\{E,F,G\}$	*Poor*
x_{37}	$\{F,G\}$	$\{F\}$	$\{F,G\}$	$\{F,G\}$	*Good*
x_{38}	$\{F,G\}$	$\{F,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_{39}	$\{E,F\}$	$\{F\}$	$\{E,F,G\}$	$\{F\}$	*Poor*
x_{40}	$\{F\}$	$\{E,F\}$	$\{F\}$	$\{G\}$	*Good*
x_{41}	$\{E,F,G\}$	$\{F,G\}$	$\{F\}$	$\{E,F,G\}$	*Poor*
x_{42}	$\{E,F,G\}$	$\{E,F,G\}$	$\{E,F,G\}$	$\{E,G\}$	*Poor*
x_{43}	$\{E,F\}$	$\{F\}$	$\{E,F,G\}$	$\{F\}$	*Poor*
x_{44}	$\{F\}$	$\{F\}$	$\{F,G\}$	$\{E,F\}$	*Good*
x_{45}	$\{E,F,G\}$	$\{E,F,G\}$	$\{F,G\}$	$\{E,F,G\}$	*Good*
x_{46}	$\{E,G\}$	$\{E,G\}$	$\{GF\}$	$\{F,G\}$	*Good*
x_{47}	$\{E,F,G\}$	$\{E,F,G\}$	$\{E,F,G\}$	$\{E,G\}$	*Poor*
x_{48}	$\{E,F\}$	$\{E,G\}$	$\{F,G\}$	$\{F\}$	*Poor*
x_{49}	$\{E,F,G\}$	$\{F,G\}$	$\{E,F,G\}$	$\{E,G\}$	*Poor*
x_{50}	$\{F,G\}$	$\{F\}$	$\{F,G\}$	$\{F,G\}$	*Good*

Table 15
The discernibility matrix of Table 14.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	...	...	...	x_{48}	x_{49}	x_{50}
x_1	∅	A, S, W	A, S	∅	∅	...	...	...	∅	∅	A, S
x_2	∅	∅	∅	∅	∅	...	...	...	∅	R	∅
x_3	A, S	∅	∅	A, S	S	...	...	...	S	A, S, R, W	∅
x_4	∅	A, S, R	A, S, W	∅	∅	...	...	...	S, W	A, R, S, W	A, S, W
x_5	∅	A, S, W	W	∅	∅	...	...	...	S, W	A, R, W	W
⋮						⋱					
⋮							⋱				
⋮								⋱			
x_{48}	∅	A, S, R	A, S, W	∅	∅	...	...	...	∅	∅	A, S, W
x_{49}	∅	S, W	W	∅	∅	...	...	...	∅	∅	W
x_{50}	A, S	∅	∅	A, S	S	...	...	...	S	A, S, R, W	∅

5.3. Experimental analysis

In many practical decision-making issues, set-valued information systems and set-valued decision tables have very wide applications, which can be used in intelligent decision-making and knowledge discovery from information systems with uncertain information and set-valued information [53,54]. A decision-maker may need to adopt a better one from some possible projects or find some directions from existing successful projects before making a decision. The purpose of this section is to illustrate how to obtain criteria reducts from set-valued ordered information systems and set-valued ordered decision tables by using the approaches proposed in this paper.

Simply, we only deal with a real-world conjunctive set-valued ordered information system for this target. We omit the criteria reduction from disjunctive set-valued ordered information systems in this section.

Let us consider a practical decision issue from the test for foreign language ability in Shanxi University. The test results can be expressed as a conjunctive set-valued information system. Test factors are classified into four factors, which are Audition, Spoken language, Reading and Writing. These four factors are all inclusion increasing preferences and the value of each student under each factor is given by an evaluation expert through a set-value. The test results is shown as Table 12, which can be downloaded in [55], where $U = \{x_1, x_2, x_3, \ldots, x_{49}, x_{50}\}$. For convenience, in the sequel, A, S, R, W will stand for Audition, Spoken language, Reading and Writing, respectively.

From Table 12, through using criteria reduction to conjunctive set-valued OIS in Section 5.1, one can obtain its discernibility matrix, which is shown in Table 13 (its detailed description is put in [55]). Hence, from this discernibility matrix of Table 13, one can obtain

$$M_\wedge = (A \vee S \vee W) \wedge (A \vee S) \wedge (A \vee S \vee R \vee W) \wedge R \wedge (A \vee R \vee W) \wedge S \wedge W \wedge (S \vee R \vee W) \wedge (A \vee W) \wedge (R \vee W) \wedge (A \vee S \vee R) \wedge (A \vee R) \wedge (S \vee W) \wedge (S \vee R) = S \wedge R \wedge W.$$

Therefore, $\{S, R, W\}$ is a unique criteria reduct for this conjunctive set-valued ordered information system. In other words, the attribute Audition can be reduced from Table 12.

To extract concise dominance rules, we need to compute relative criteria reduction of an ordered decision tables. In what follows, we analyze the relative criteria reduction from a conjunctive set-valued ordered decision table. Table 14 is one of this kind of decision tables, which denotes test results about language ability and also can be downloaded in [55], where $U = \{x_1, x_2, x_3, \ldots, x_{49}, x_{50}\}$. For convenience, in the sequel, A, S, R, W and E will stand for the condition attributes Audition, Spoken language, Reading, Writing and the decision attribute Evaluation, respectively.

Through using the relative criteria reduction to conjunctive set-valued ODT in Section 5.1, one can derive its discernibility matrix from Table 14, which is shown in Table 15. Hence, from this discernibility matrix of Table 15, it can be calculated that

$$M^*_\wedge = (A \vee S) \wedge (A \vee S \vee R \vee W) \wedge R \wedge (A \vee S \vee W) \wedge (A \vee R \vee W) \wedge S \wedge W \wedge (S \vee R \vee W) \wedge (A \vee W) \wedge (R \vee W) \wedge (A \vee S \vee R) \wedge (A \vee R) \wedge (S \vee W) \wedge (S \vee R) = S \wedge R \wedge W.$$

Thus, there is a relative criteria reduct in this ordered decision table, which is {Spoken language, Reading, Writing}. From this result, we learn that the test factor Audition is indispensable for this decision problem. In other words, these three factors Spoken language, Reading and Writing are three important factors for evaluating the language ability of each student. Through this relative criteria reduct, one can obtain a set of more briefer dominance rules from the original ordered decision table.

6. Conclusions

Set-valued information systems are generalized models of single-valued information systems, and can be classified into two categories: disjunctive and conjunctive. We deal with two types of set-valued information systems and decision tables in present research. Based on the relation between set-values, in this paper, we have introduced dominance relations $R_A^{\wedge\geqslant}$

and $R_A^{\vee\geqslant}$ to conjunctive set-valued information systems and disjunctive set-valued information systems, respectively, and have given a ranking method for all objects through using dominance degree of each object. Based on these two dominance relations, we have established a rough set approach in these types of OIS, which is mainly based on the substitution of the indiscernibility relation by the dominance relations. In addition, we have also discussed conjunctive set-valued ordered decision tables, disjunctive set-valued ordered decision tables, and decision rules extracted from these two types of ordered decision tables. In order to extract concise decision rules, through using discernibility matrices, we have proposed criteria reductions to these two types of set-valued ordered information systems and decision tables, which eliminate those information that are not essential from the view of the ordering of objects or decision rules. The approaches show how to simplify a conjunctive/disjunctive set-valued ordered information system and find decision rules directly from a conjunctive/disjunctive set-valued ordered decision table.

Acknowledgements

The authors wish to thank the anonymous reviewers for their constructive comments. Also, we would like to thank Professor Witold Pedrycz for useful comments on this study.

This work was supported by the national natural science foundation of China (No. 60773133, No. 60573074), the high technology research and development program of China (No. 2007AA01Z165), the national key basic research and development program of China (973) (No. 2007 CB311002) and the natural science foundation of Shanxi province (No. 2008011038, No. 2007021015).

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